國立政治大學 111 學年度第 1 學期博士班資格考考試命題紙

考試科目	理論統計	卷別	第 1 卷	考試日期	111 年 9 月 5 日 星期一

1. (20 points)
   (a) Show that $X_n \to X$ w.p.1 if and only if $P(|X_n - X| \geq \epsilon \text{ i.o.}) = 0$ for each $\epsilon > 0$.
   (b) Use (a) to show that convergence w.p.1 implies convergence in probability.

2. (20 points) Let X be a simple random variables. Let $\mathcal{G}$ be a σ-field.
   (a) Show that, if $\mathcal{G} = \{\varnothing, \Omega\}$, then X is measurable $\mathcal{G}$ if and only if X is constant.
   (b) Suppose that $X = \sum_{i=1}^{5} a_i 1_{A_i}$, where A_i's are disjoint and $a_1 = -1, a_2 = 0, a_3 = 1, a_4 = 2, a_5 = 3$. Let $Y = X^2$. Find $\sigma(Y)$, the σ-field generated by Y, and argue that $\sigma(Y) \subset \sigma(X)$.

3. (20 points)
   (a) What does it mean that $\{f_n\}$ is uniformly integrable?
   (b) Show that if $\{f_n\}$ is uniformly integrable, then $\int |f_n| < M$, for some $M < \infty$.

4. (20 points) Consider a stochastic process $\{X_n, n = 0, \pm 1, \pm 2, ...\}$. Suppose that $X_n = \alpha X_{n-1} + e_n$, $n = 0, \pm 1, \pm 2, ...$, where $0 < \alpha < 1$, e_n's are i.i.d. N(0,1), and e_n is independent of X_{n-1}. Suppose that it is known that the stationary distribution of the process is $N(0, \sigma^2)$; that is, $X_n \sim N(0, \sigma^2)$.
   (a) Express σ^2 in terms of α.
   (b) Is $\{X_n, n \geq 1\}$ uniformly integrable? Justify your answer.

5. (10 points) Consider a game where there are n closed doors. Behind one of these doors is a car; behind the other $n-1$ are goats. The player does not know where the car is, but the host does. The player picks a door and the host opens one of the remaining doors which does not hide the car. After the host has shown a goat behind the door, the player is given the option to switch doors. What is the probability of winning the car if the player decides to switch?

6. (10 points) Consider independent tosses of a coin where, on each toss, the probability of obtaining a head (H) is p and the probability of obtaining a tail (T) is $q = 1 - p$. What is the expected number of tosses needed for the pattern HH to appear?

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考試科目	理論統計	卷別	第2卷	考試日期	111年9月5日 星期一

- Notation.
  - $R = (-\infty, \infty)$.
  - $\mathcal{B}(R)$ denotes the Borel σ-field on R.
  - For $a \in R$, δ_a denotes the measure on $(R, \mathcal{B}(R))$ such that for $A \in \mathcal{B}(R)$,
  $$\delta_a(A) = \begin{cases} 1 & \text{if } a \in A; \\ 0 & \text{if } a \notin A. \end{cases}$$
  - For a measurable space $(\Omega, \mathcal{F})$ and two measures μ_1, μ_2 on $(\Omega, \mathcal{F})$, $\mu_1 + \mu_2$ denotes the measure on $(\Omega, \mathcal{F})$ such that for $A \in \mathcal{F}$,
  $$(\mu_1 + \mu_2)(A) = \mu_1(A) + \mu_2(A).$$
- The problems are given below. Total points: 100 pts.
  1. (10 pts) Suppose that μ_1 and μ_2 are measures on a measurable space $(\Omega, \mathcal{F})$. Suppose that f is a measurable function from $(\Omega, \mathcal{F})$ to $(R, \mathcal{B}(R))$ and f is integrable with respect to $\mu_1 + \mu_2$. Show that
  $$\int f d(\mu_1 + \mu_2) = \int f d\mu_1 + \int f d\mu_2.$$
  2. (5 pts) Suppose that $f(x) = e^{-x^2/2}$ for $x \in R$ and
  $$g(x) = \begin{cases} f(x) & \text{if } x \neq 1; \\ 0 & \text{if } x = 1. \end{cases}$$
  Let λ be the Lebesgue measure on $(R, \mathcal{B}(R))$. Show that
  $$\int f d(\lambda + \delta_1) \neq \int g d(\lambda + \delta_1).$$
  3. (55 pts) Suppose that k is a positive integer and $c_1, \ldots, c_k$ are k constants in R. Suppose that U, $Z_1, \ldots, Z_k$ are independent random variables on a probability space $(\Omega, \mathcal{F}, P)$. Suppose that $\sum_{j=0}^{k} P(U = j) = 1$ and for $j \in \{1, \ldots, k\}$, $Z_j \sim N(c_j, 1)$. Define
  $$X = \begin{cases} 1 & \text{if } U = 0; \\ Z_j & \text{if } U = j \text{ and } j \in \{1, \ldots, k\}. \end{cases}$$
  Let λ denote the Lebesgue measure on $(R, \mathcal{B}(R))$ and let $\pi_j = P(U = j)$ for $j \in \{0, 1, \ldots, k\}$.

(a) (20 pts) Find a density of X with respect to $\lambda + \delta_1$.

(b) (20 pts) Does the distribution of X have a density with respect to the measure $\lambda + \delta_1 + \delta_2$? If so, give a density of X with respect to $\lambda + \delta_1 + \delta_2$. If not, explain why.

(c) (15 pts) Find $E(U|X)$. You may use $\pi_0, \ldots, \pi_k$ and $c_1, \ldots, c_k$ in the expression of $E(U|X)$.

4. (20 pts) Suppose that $\{X_n\}_{n=1}^{\infty}$ is a sequence of independent random variables on the same probability space and

$$P\left(X_i = \frac{1}{i}\right) = 0.5 = P\left(X_i = \frac{i-1}{i}\right)$$

for $i = 1, \ldots, n$. Let $S_n = \sum_{i=1}^{n} X_i$ for $n \geq 1$. Show that

$$\frac{S_n - 0.5n}{\sqrt{Var(S_n)}}$$

converges to $N(0,1)$ in distribution as $n \to \infty$.

5. (10 pts) Suppose that $(X_1, \ldots, X_n)$ is a sample and $X_1, \ldots, X_n$ are independent and identically distributed. Suppose that the distribution of X_1 is the same as the distribution of the X in Problem 1 with $k = 1$. Propose an estimator of (π_1, c_1) so that the proposed estimator converges to (π_1, c_1) in probability as $n \to \infty$. Justify your answer.

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考試科目	理論統計	卷別	第 3 卷	考試日期	111 年 9 月 5 日 星期一

1. Consider the multiple linear regression model $\mathrm{Y} = \mathrm{X}\beta + \epsilon$ with X being a $n \times p$ full rank matrix and $\epsilon \sim N(0, \sigma^2 \mathrm{I})$. Let $\widehat{\beta}$ and e be the least squares estimator (LSE) of β and the residual, respectively.

   (a) (10%) Derive the distribution of $\widehat{\beta}$ and $SSE = \mathrm{e}^\top \mathrm{e}$ and show that both of them are independent.

   (b) (10%) State and prove the best linear unbiased estimator $\widehat{\beta}$.

   (c) (10%) For $\alpha \in (0, 1)$, construct a $100(1-\alpha)\%$ confidence interval for $\mathrm{x}_0^\top \beta$ and a $(1-\alpha)$ prediction interval for the future independent run $Y_0 = \mathrm{x}_0^\top \beta + \epsilon_0$ with $\epsilon_0 \sim (0, \sigma^2)$.

2. Consider the linear model

$$\mathrm{Y} = \mathrm{X}\beta + \mathrm{Z}\gamma + \epsilon \quad \text{with} \quad \epsilon \sim N(0, \sigma^2 \mathrm{I}),$$

   where $\mathrm{Y}, \epsilon \in \mathbb{R}^n$, $\mathrm{X} \in \mathbb{R}^{n \times p}$, $\mathrm{Z} \in \mathbb{R}^{n \times q}$, $\beta \in \mathbb{R}^p$, and $\gamma \in \mathbb{R}^q$. Let $\widehat{\beta}$ denote the LSE of β.

   (a) (10%) Find the LSE $\widehat{\beta}$.

   (b) (10%) Compute $\mathrm{var}(\widehat{\beta})$ and find the asymptotic distribution.

   (c) (10%) Derive the suitable test statistic for the hypothesis test $\begin{cases} H_0 : \beta = 0 \\ H_1 : \beta \neq 0 \end{cases}$.

   (d) (10%) Consider a special case that $p = q = 1$ with $\beta, \gamma \in \mathbb{R}$. For $i = 1, \cdots, n$, let X_i and Z_i denote univariate covariates with $E(X_i) = E(Z_i) = 0$, $\mathrm{var}(X_i) = \sigma_X^2$, and $\mathrm{var}(Z_i) = \sigma_Z^2$. Let Y_i be a univariate response. Suppose that X_i is contaminated with measurement error while Z_i is precisely measured. Let X_i^* denote the surrogate version of X_i. The following classical measurement error model is usually adopted to characterize X_i and X_i^*:

$$X_i^* = X_i + \eta_i \quad \text{with} \quad \eta_i \sim (0, \sigma_\eta^2). \tag{1}$$

   Assume that η_i is independent of X_i and Y_i. Let $\widetilde{\beta}$ denote the resulting LSE based on (1). Please examine if $\widetilde{\beta}$ is a biased estimator.

3. Consider the binary response $Y \in \{0, 1\}$ and the p-dimensional vector of covariates X.

   (a) (5%) Characterize the relationship between Y and X by a logistic regression model with the associated parameter β.

   (b) (5%) Let $\{(\mathrm{X}_i, Y_i) : i = 1, \cdots, n\}$ denote the random sample with size n. Write the corresponding likelihood function and the estimation procedure for the parameter β.

   (c) (10%) Demonstrate an algorithm to derive the estimator of the parameter for the logistic regression model.

(d) (10%) Suppose that β is sparsity in the sense that some components are zero. Please describe regularization methods to do variable selection and estimation. In addition, explain oracle properties.