國立政治大學 110 學年度第 1 學期博士班資格考考試命題紙

考試科目	理論統計	卷 別	第一卷	考試日期	110 年 9 月 6 日 星期一

數理統計、機率論（總分：50）

1. (30 pts) Suppose that we have a sample of IID observations $X_1, \ldots, X_n$ and the X_i's have a common Lebesgue density function f. Suppose that f is twice differentiable on $(-\infty, \infty)$ and f'' is continuous and bounded on $(-\infty, \infty)$. For $x_0 \in (-\infty, \infty)$, consider the following estimator of $f(x_0)$:

$$\hat{f}_n(x_0) = \frac{1}{nh_n} \sum_{i=1}^{n} k\left(\frac{x_0 - X_i}{h_n}\right),$$

where $\{h_n\}_{n=1}^{\infty}$ is a sequence of real numbers such that $\lim_{n\to\infty} h_n = 0$ and

$$k(x) = \begin{cases} 1 - |x| & \text{if } |x| \le 1; \\ 0 & \text{otherwise.} \end{cases}$$

   (a) (10 pts) Find

$$\lim_{n\to\infty} \frac{E(\hat{f}_n(x_0)) - f(x_0)}{h_n^2}.$$

   (b) (12 pts) Specify condition(s) on $\{h_n\}_{n=1}^{\infty}$ to guarantee that for $x_0 \in (-\infty, \infty)$, $\hat{f}_n(x_0)$ converges to $f(x_0)$ in probability as $n \to \infty$. Jusifty your answer.

   (c) (8 pts) Show that

$$\frac{\hat{f}_n(x_0) - E(\hat{f}_n(x_0))}{\sqrt{Var(\hat{f}_n(x_0))}}$$

   converges to $N(0,1)$ in distribution as $n \to \infty$. Specify additional condition(s) if necessary.

2. (10 pts) Suppose that T, T_j: $j = 1, 2, \ldots$ are decision rules for the same decision problem. Suppose that for $j \ge 1$, T_j is a Bayes rule with Bayes risk r_j. Suppose that the risk function of T is a constant $r \in (-\infty, \infty)$ and

$$r = \lim_{j\to\infty} r_j.$$

Show that T is a minimax decision rule.

3. (10 pts) Suppose that we have a sample of IID observations $X_1, \ldots, X_n$, where the distribution of X_1 is $N(\mu, 1)$: the normal distribution of mean μ and variance 1, and $\mu \in (-\infty, \infty)$ is an unknown parameter. Consider the problem of estimating μ under a loss function L, where L is defined so that the loss for estimating μ using a when $\mu = \mu_0$ is

$$L(a, \mu_0) = (a - \mu_0)^2$$

for $a \in (-\infty, \infty)$ and $\mu_0 \in (-\infty, \infty)$.

   (a) (5 pts) Find the Bayes estimator for μ under the loss L when the prior distribution for μ is $N(0, \tau^2)$: the normal distribution of mean zero and variance τ^2, where τ is a positive constant.

   (b) (5 pts) Find a minimax estimator for μ.

考試科目	理論統計	卷別	第二卷	考試日期	110年9月6日 星期一

線性模式（總分：50）

NOTATION:

(i) $\mathbf{0}_n$: the zero n-vestors in $\mathcal{R}^n$.

(ii) I_n: the identity matrix of size n.

(iii) P_V: the projection matrix on a vector space V.

(iv) $\mathcal{L}(X)$: the subspace spanned by the column vectors of a matrix X.

1. For the linear model $\mathbf{Y} = X\boldsymbol{\beta} + \boldsymbol{\epsilon}$, where X is an $n \times p$ matrix and $\boldsymbol{\epsilon}$ has mean 0 and covariance matrix $\sigma^2 I_n$, we say $\eta = \mathbf{c}'\boldsymbol{\beta}$, $\mathbf{c} \in \mathcal{R}^p$, is estimable if and only if there exists a vector $\mathbf{a}$ in $\mathcal{R}^n$ such that $E(\mathbf{a}'\mathbf{Y}) = \mathbf{c}'\boldsymbol{\beta}$ for all $\boldsymbol{\beta} \in \mathcal{R}^p$.

   (a) (5pts). Please show that $\mathbf{c}'\boldsymbol{\beta}$ is estimable if and only if $\mathbf{c}$ lies in the row space of X. (Therefore, all components of $\boldsymbol{\beta}$ are estimable if and only if the column vectors of X are linear independent.)

   (b) (10pts). Let $\mathbf{a}_* = P_{\mathcal{L}(X)}\mathbf{a}$. Please show that $\mathbf{a}_*'\mathbf{Y}$ has the smallest variance among all linear unbiased estimators of η. Namely, $\mathbf{a}_*'\mathbf{Y}$ is unbiased and $var(\mathbf{a}_*'\mathbf{Y}) \leq var(\mathbf{d}'\mathbf{Y})$ for all $E(\mathbf{d}'\mathbf{Y}) = \eta$.

2. (7pts). Consider the regression model $\mathbf{Y} = X\boldsymbol{\beta} + \boldsymbol{\epsilon}$ with $E(\boldsymbol{\epsilon}) = \mathbf{0}_n$ and $Cov(\boldsymbol{\epsilon}) = \sigma^2 I_n$. Let us decompose $\mathbf{Y} = \begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{pmatrix}$ and $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$, where $\mathbf{Y}_i$ and X_i have n_i rows for $i = 1, 2$ and $n = n_1 + n_2$. Let SSE, SSE_1 and SSE_2 denote the usual error sum of squares for regression of $\mathbf{Y}$ on X, $\mathbf{Y}_1$ on X_1 and $\mathbf{Y}_2$ on X_2 respectively. Namely, SSE$= ||\mathbf{Y} - P_{\mathcal{L}(X)}\mathbf{Y}||^2$, $\text{SSE}_1 = ||\mathbf{Y}_1 - P_{\mathcal{L}(X_1)}\mathbf{Y}_1||^2$, and $\text{SSE}_2 = ||\mathbf{Y}_2 - P_{\mathcal{L}(X_2)}\mathbf{Y}_2||^2$. Which one of the following statements is correct? Justify your answer.

   (i) $\text{SSE} \leq \text{SSE}_1 + \text{SSE}_2$; (ii) $\text{SSE} = \text{SSE}_1 + \text{SSE}_2$; (iii) $\text{SSE} \geq \text{SSE}_1 + \text{SSE}_2$;

   (iv) none of the above.

3. Here we consider two linear regression models as below.

   Model (A): $\mathbf{Y} = X_1\boldsymbol{\beta}_1 + \boldsymbol{\epsilon}$;

   Model (B): $\mathbf{Y} = X_1\boldsymbol{\beta}_1 + X_2\boldsymbol{\beta}_2 + \boldsymbol{\delta}$.

   Here X_1 and X_2 are $n \times p_1$ and $n \times p_2$ constant matrices, $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$ are unknown p_1-vector and p_2-vector, and $\boldsymbol{\epsilon}$, $\boldsymbol{\delta}$ are random vectors with mean $\mathbf{0}_n$ and covariance matrix $\sigma^2 I_n$. Suppose matrix $X_{n\times(p_1+p_2)} = [X_1 \quad X_2]$ has full column rank. Denote $\hat{\boldsymbol{\beta}}_1$ the unique LSE under model (A) and $\tilde{\boldsymbol{\beta}}_1$ and $\tilde{\boldsymbol{\beta}}_2$ the unique LSE under model (B). Also let $\hat{\sigma}^2 = \frac{\|\mathbf{Y}-X_1\hat{\beta}_1\|^2}{n-p_1}$ and $\tilde{\sigma}^2 = \frac{\|\mathbf{Y}-X_1\tilde{\beta}_1-X_2\tilde{\beta}\|^2}{n-p_1-p_2}$ be the usual estimates of σ^2 based on model (A) and (B) respectively.

   (a) (8pts). If model (A) is true, it is known that $\hat{\boldsymbol{\beta}}_1$ and $\hat{\sigma}^2$ are both unbiased. What about $\tilde{\boldsymbol{\beta}}_1$ and $\tilde{\sigma}^2$? Justify your answer.

   (b) (10pts). If model (B) is true, we know $\tilde{\boldsymbol{\beta}}_1$ and $\tilde{\sigma}^2$ are unbiased. However, $\hat{\boldsymbol{\beta}}_1$ and $\hat{\sigma}^2$ are not. Please find bias for both of them (in terms of $X_1, X_2, \boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \sigma^2$, ...).

   (c) (10pts). Suppose model (B) is true. Please find a necessary and sufficient condition (on X_1 and X_2) such that $\hat{\boldsymbol{\beta}}_1$ is unbiased. Justify your answer. Please also make a comment onthe bias of $\hat{\sigma}^2$ under this condition.