國立政治大學 105 學年度第 2 學期博士班資格考考試命題紙

考試科目	數理統計專題	卷別	第一卷	考試日期	106年2月13日 星期一

1. Suppose that $(X_1, \ldots, X_n)$ is a sample of IID observations, and X_1 has a density function f with respect to the Lebesgue measure on $(-\infty, \infty)$. Suppose that there exists a unique number μ such that $P(X_1 < \mu) = 0.5$ and $P(X_1 > \mu) = 0.5$.

   (a) (5 points) Suppose that $\hat{\mu}$ is an estimator for μ based on $(X_1, \ldots, X_n)$ such that

   $$\lim_{n\to\infty} P(\hat{\mu} > \mu + \delta) = 0 = \lim_{n\to\infty} P(\hat{\mu} < \mu - \delta)$$

   for $\delta > 0$. Show that $\hat{\mu}$ converges to μ in probability as $n \to \infty$.

   (b) (15 points) Let $\hat{\mu}$ be the sample median of $(X_1, \ldots, X_n)$. Show that $\hat{\mu}$ converges to μ in probability as $n \to \infty$.

2. Suppose that $(X_1, \ldots, X_n)$ is a random sample, and X_1 has the same distribution as σZ, where $\sigma > 0$ is an unknown parameter and the density function of Z with respect to the Lebesgue measure on $(-\infty, \infty)$ is given by

   $$f(z) = c_0 e^{-z^3} \text{ for } z \in (0, \infty),$$

   where $c_0 = 1/\int_0^\infty e^{-z^3} dz$.

   (a) (10 points) Find the UMVUE for σ based on $(X_1, \ldots, X_n)$.

   (b) (5 points) Consider the problem of estimating σ under the loss function $L(\sigma, a) = (a/\sigma - 1)^2$ for $a > 0$, $\sigma > 0$. Show that the problem is invariant under the group of scale transforms

   $$\{T_c : c > 0, T_c(x_1, \ldots, x_n) = (cx_1, \ldots, cx_n) \text{ for } (x_1, \ldots, x_n) \in R^n\}.$$

   (c) (5 points) Is the estimator found in Part (a) invariant under the scale transform with the loss function in Part (b)? Justify your answer.

3. Suppose that $(X_1, \ldots, X_n)$ is a random sample from the beta distribution $beta(\alpha, \beta)$, which has a density function f with respect to the Lebesgue measure on $(-\infty, \infty)$, where

   $$f(x) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} & \text{if } x \in (0,1); \\ 0 & \text{otherwise,} \end{cases}$$

   where α and β are positive parameters and Γ denotes the gamma function, which is defined by $\Gamma(a) = \int_0^\infty t^{a-1} e^{-t} dt$ for $a > 0$.

   (a) (5 points) Suppose that $\beta = 1$. Find a statistic U that is complete and sufficient for α.

   (b) (5 points) Let $S = \sum_{i=1}^n \ln(X_i)$. Show that $(\ln(X_1)/S, \ldots, \ln(X_{n-1})/S)$ and S are independent when $\beta = 1$.

   (c) (10 points) For $\alpha^* \in (0, 1)$, find a UMPU level α^* test for the problem

   $$H_0 : \beta \le 1 \text{ versus } H_1 : \beta > 1.$$

國立政治大學 105 學年度第 2 學期博士班資格考考試命題紙

考 試 科 目	數理統計專題	卷　　別	**第一卷**	考 試 日 期	106年2月13日 星期一

4. Suppose that $(X_1, \ldots, X_n)$ is a random sample, $n \geq 2$ and

$$P(X_1 = k) = \theta(1-\theta)^k \text{ for } k = 0, 1, 2, \ldots,$$

where $\theta \in (0,1)$. Consider the problem of estimating $(1-\theta)/\theta$. For $\theta \in (0,1)$ and $a > 0$, the loss for estimating $(1-\theta)/\theta$ by a is

$$\theta^{a_0}(1-\theta)^{b_0}\left(\frac{1-\theta}{\theta} - a\right)^2, \tag{1}$$

where a_0 and b_0 are given constants.

(a) (5 points) Show that for a nonnegative integer s,

$$P\left(\sum_{i=1}^{n} X_i = s\right) = \frac{(n+s-1)!}{s!(n-1)!}\theta^n(1-\theta)^s.$$

(b) (5 points) Find the Bayes rule for estimating $(1-\theta)/\theta$ under the loss function in (1) with $(a_0, b_0) = (2, 1)$, where the prior distribution for θ is $beta(\alpha, \beta)$. The Lebesgue density for $beta(\alpha, \beta)$ is given in Problem 3.

(c) (5 points) Show that the Bayes rule found in Part (b) with $(a_0, b_0) = (2, 1)$ is admissible under the loss function in (1) with $(a_0, b_0) = (2, 1)$.

(d) (5 points) Determine whether the Bayes rule found in Part (b) with $(a_0, b_0) = (2, 1)$ is admissible under the loss function in (1) with $(a_0, b_0) = (2, -1)$.

5. Suppose that $X = (X_1, \ldots, X_n)$ and $Y = (Y_1, \ldots, Y_n)$ are two random samples, X and Y are independent, and the distributions of X and Y depend on unknown parameters θ and η respectively. Suppose that $\hat{\theta}_n$ is an estimator for θ based on X, $\hat{\eta}_n$ is an estimator for η based on Y, and as $n \to \infty$, $\sqrt{n}(\hat{\theta}_n - \theta)$ converges in distribution to the standard normal distribution and $a_n(\hat{\eta}_n - \eta)$ converges in distribution to the exponential distribution with mean 1, where $\{a_n\}_{n=1}^{\infty}$ is a positive nonrandom real sequence such that $\lim_{n\to\infty} a_n = \infty$. Suppose that $\eta > 0$ and $\theta > 0$. Consider the problem of estimating η/θ.

(a) (5 points) Show that $\hat{\theta}_n/\hat{\eta}_n$ converges to η/θ in probability as $n \to \infty$.

(b) (10 points) Suppose that $a_n = \sqrt{n}$. Show that $a_n(\hat{\theta}_n/\hat{\eta}_n - \eta/\theta)$ converges in distribution as $n \to \infty$ and find the variance of the limiting distribution.

(c) (5 points) Suppose that $\lim_{n\to\infty} \sqrt{n}/a_n = 0$. Find $\{b_n\}_{n=1}^{\infty}$: a positive nonrandom real sequence such that $\lim_{n\to\infty} b_n = \infty$ and $b_n(\hat{\theta}_n/\hat{\eta}_n - \eta/\theta)$ converges in distribution to a distribution with positive variance as $n \to \infty$. Justify your answer.

國立政治大學 105 學年度第 2 學期博士班資格考考試命題紙

考 試 科 目	數理統計專題	卷　　別	**第二卷**	考 試 日 期	106 年 2 月 13 日 星期一

1. Let $X_1, X_2, \ldots,$ be independent random variables such that X_n has the uniform distribution on $[-n, n]$ for $n = 1, 2, \ldots$.

   (a) (20pts). Please show that the Lindeberg's condition is satisfied for $\{X_{nj}, j = 1, \ldots, n\}_{n=1}^{\infty}$, where $X_{nj} = X_j$.

   (b) (10pts). Please state the resulting central limit theorem.

2. Let $\mathbf{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_k \end{bmatrix}$, $\mathbf{p} = \begin{bmatrix} p_1 \\ \vdots \\ p_k \end{bmatrix}$, and $\boldsymbol{\alpha} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_k \end{bmatrix}$. Suppose that $\mathbf{X}$ given $\mathbf{p}$ has a multinomial distribution $\mathcal{M}(n, \mathbf{p})$, and that $\mathbf{p}$ has a prior of Dirichlet distribution with parameter $\boldsymbol{\alpha}$. We say $\mathbf{p} \sim \mathcal{D}(\boldsymbol{\alpha})$ if the pdf of $\mathbf{p}$ is $\pi(\mathbf{p}) = \frac{\Gamma(\alpha_1+\cdots+\alpha_k)}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_k)} p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ when $p_1 + \cdots + p_k = 1$, $0 \leq p_i \leq 1$, and $0 < \alpha_i$, for $i = 1, 2, \ldots$. Also note, mean of p_i is $\frac{\alpha_i}{\alpha_1+\cdots+\alpha_k}$.

   (a) (15pts). What is the posterior distribution of $\mathbf{p}$ given $\mathbf{X}$?

   (b) (15pts). Please give the Bayes estimator of $\mathbf{p}$ under the squared loss function $L(\hat{\mathbf{p}}, \mathbf{p}) = ||\hat{\mathbf{p}} - \mathbf{p}||^2 = \Sigma_{i=1}^{k} (\hat{p}_i - p_i)^2$.

考試科目	數理統計專題	卷別	**第二卷**	考試日期	106年2月13日 星期一

3. Suppose $\{X_i\}_{i=1}^n$ are independent random variables and $X_i \sim \mathcal{N}(\mu_i, 1)$. To estimate $\boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_n \end{bmatrix}$ using $\mathbf{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}$, it has been shown that, for $n > 2$, the Stein-James estimator $T^*(\mathbf{X}) = (1 - \frac{n-2}{\Sigma_{i=1}^n X_i^2})\mathbf{X}$ dominates the estimator $T(X) = \mathbf{X}$ under the squared loss function $L(\hat{\boldsymbol{\mu}}, \boldsymbol{\mu}) = ||\hat{\boldsymbol{\mu}} - \boldsymbol{\mu}||^2 = \Sigma_{i=1}^n (\hat{\mu}_i - \mu_i)^2$. The proof follows from the two lemmas below.

   Lemma 1: Let $g : \mathcal{R} \longrightarrow \mathcal{R}$ be a differentiable function. Suppose X has a normal distribution with mean μ and variance 1, and $\mathrm{E}(|g'(X)|) < \infty$. Then $\mathrm{E}(|g'(X)|) = Cov(X, g(X))$.

   Lemma 2: Let $g = (g_1, \ldots, g_n) : \mathcal{R}^n \longrightarrow \mathcal{R}^n$ be a vector of differentiable functions. Suppose $\mathbf{X}$ has a multivariate normal distribution with mean vector $\boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_n \end{bmatrix}$ and covariance matrix $\mathbf{I}_n = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}$, and $\mathrm{E}(|h_i(X_i)|) < \infty$, where $h(y)$ $=\mathrm{E}(\frac{\mathrm{d}}{\mathrm{d}y} g_i(X_1, \ldots, X_{i-1}, y, X_{i+1}, \ldots, X_n))$. Then $\mathrm{E}(||\mathbf{X} + g(\mathbf{X}) - \boldsymbol{\mu}||^2) = n + \mathrm{E}(||g(\mathbf{X})||^2) + 2\mathrm{E}(\Sigma_{i=1}^n \frac{\partial}{\partial x_i} g_i(\mathbf{X}))$.

   (a) (20pts). Please prove Lemma 1. (Hint: $\mathrm{E}(g'(X)) = \int_{-\infty}^{\infty} g'(x)\phi(x-\mu)\mathrm{d}x = \int_{-\infty}^{0} g'(x)\phi(x-\mu)\mathrm{d}x + \int_{0}^{\infty} g'(x)\phi(x-\mu)\mathrm{d}x$ and $\phi(x-\mu) = -\int_{-\infty}^{x}(t-\mu)\phi(t-\mu)dt = \int_{x}^{\infty}(t-\mu)\phi(t-\mu)dt$, where ϕ is the standard normal density function. Also note $\mathrm{Cov}(X, g(X)) = \mathrm{E}((X-\mu)g(X))$.)

   (b) (20pts). Please prove Lemma 2. (Hint: by the fact $\mathrm{E}(||\mathbf{X} - \boldsymbol{\mu}||^2) = n$ and Lemma 1.)

國立政治大學 105 學年度第 2 學期博士班資格考考試命題紙

考 試 科 目	數理統計專題	卷 別	第三卷	考 試 日 期	106 年 2 月 13 日 星期一

Please show all your work to get the full credit.

1. (5 pts) Consider a sample of size n from the exponential distribution with mean λ. Find a $(1-\alpha)\times 100\%$ confidence interval for λ using the pivot quantities.

2. Let $X_1, \ldots, X_n$ be a random sample from the Pareto distribution with pdf

$$f(x|\lambda, \theta) = \theta\lambda^{\theta}x^{-(\theta+1)}, \quad x > \lambda,$$

and two unknown parameters $\lambda > 0$ and $\theta > 2$.

(a) (5 pts) Find the MLEs of λ and θ.

(b) (5 pts) Find the method of moments estimators of λ and θ.

3. Let $X_1, \ldots, X_n$ be a random sample from a normal distribution $N(\mu, \sigma^2)$. Consider testing

$$H_0 : \theta \le \theta_0 \quad \text{versus} \quad H_1 : \theta > \theta_0$$

Let $\bar{X}_m$ denote the sample mean of the first m observation $X_1, \ldots, X_m$, for $m = 1, \ldots, n$.

(a) (5 pts) Find the *uniformly minimum variance unbiased estimator* (UMVUE) of μ^2.

(b) (5 pts) Please state the definition of a *uniformly most powerful* (UMP) level α test.

(c) (6 pts) If σ^2 is known, show that the test that rejects H_0 when

$$\bar{X}_n > \theta_0 + z_{\alpha}\sqrt{\sigma^2/n}$$

is a test of size α.

(d) (6 pts) Show that the test in part (c) can be derived as an likelihood ratio test (LRT).

(e) (6 pts) Show that the test in part (c) is a UMP test.

(f) (6 pts) If σ^2 is unknown, show that the test that rejects H_0 when

$$\bar{X}_n > \theta_0 + z_{n-1,\alpha}\sqrt{S^2/n}$$

is a test of size α and the test can also be derived as an LRT.

考 試 科 目	數理統計專題	卷　　別	**第三卷**	考 試 日 期	106 年 2 月 13 日 星期一

(g) (6 pts) If σ^2 is known, show that for each $m = 1, \ldots, n$, the test that rejects H_0 when

$$\bar{X}_m > \theta_0 + z_\alpha \sqrt{\sigma^2/m}$$

is an unbiased size α test.

4. Suppose that the random variable $Y_1, \ldots, Y_n$ satisfy

$$Y_i = \beta x_i + \epsilon_i, \quad i = 1, \ldots, n,$$

where $x_1, \ldots, x_n$ are fixed constants, and $\epsilon_1, \ldots, \epsilon_n$ are i.i.d. from $N(0, \sigma^2)$, σ^2 is unknown.

(a) (5 pts) Find the sufficient statistic for (β, σ^2).

(b) (5 pts) Find the MLE of β, and show that it is an unbiased estimator of β.

(c) (5 pts) Find the sampling distribution of the MLE of β.

(d) (5 pts) Show that $\sum Y_i / \sum x_i$ is an unbiased estimator of β.

(e) (5 pts) Find the exact variance of $\sum Y_i / \sum x_i$ and compare it to the variance of the MLE.

5. Let $X_1, \ldots, X_n$ be iid observations from a location-scale family. Let $T_1(X_1, \ldots, X_n)$ and $T_2(X_1, \ldots, X_n)$ be two statistics that both satisfy

$$T_i(ax_1 + b, \ldots, ax_n + b) = aT_i(x_1, \ldots, x_n)$$

for all values of $x_1, \ldots, x_n$ and b and for any $a > 0$.

(a) (5 pts) Show that T_1/T_2 is an ancillary statistic.

(b) (5 pts) Let R be the sample range and S be the sample standard deviation. Verify that R and S satisfy the above condition so that R/S is an ancillary statistic.

6. (a) (5 pts) Find the $1-\alpha$ confidence set for a by inverting the LRT of $H_0 : a = a_0$ versus $H_1 : a \neq a_0$ based on a random sample $X_1, \ldots, X_n$ from the normal distribution $N(\theta, a\theta)$, where θ is unknown.

(b) (5 pts) If a random sample $X_1, \ldots, X_n$ are from the normal distribution $N(\theta, a\theta^2)$, where θ is unknown, find the $1-\alpha$ confidence set based on inverting the LRT of $H_0 : a = a_0$ versus $H_1 : a \neq a_0$.