國立政治大學 102 學年度博士班資格考考試命題紙

考試科目	線性模式專題	卷別	第一卷	考試日期	102年9月9日 星期一

1. Let V be a subspace of $\mathcal{R}^n$ and y be a vector in $\mathcal{R}^n$. The projection of y on V is the vector $p(y|V)$ (or $P_V y$) satisfying (α): $p(y|V) \in V$ and (β): $(y - p(y|V)) \perp V$. It can be derived that the projection exists and is unique. Furthermore, if $v_1, v_2, \ldots, v_q$ in $\mathcal{R}^n$ span V and are linear independent, then $X'X$ is invertible and $P(y|V) = X(X'X)^{-1}X'y$, where ith column of X is v_i, $i = 1, 2, \ldots, q$.

   (a) (16pts). Suppose Y is a random vector in $\mathcal{R}^n$ with mean μ and covariance matrix $\sigma^2 I_n$. Please find $E(||p(Y|V)||^2)$ (in terms of μ, σ^2, X, q,...).

   Now consider the two linear regression models (A): $Y = X_1\beta_1 + \epsilon$ and (B): $Y = X_1\beta_1 + X_2\beta_2 + \epsilon$, where X_i is $n \times q_i$ of rank q_i, β_i is $q_i \times 1$, $i = 1, 2$, and ϵ has mean 0_n and covariance matrix $\sigma^2 I_n$. Denote $\hat{\sigma}^2_A$ and $\hat{\sigma}^2_B$ the usual least squares estimates of σ^2 based on model (A) and (B) respectively. Assume rank of $X = (X_1, X_2)$ is $q_1 + q_2$.

   (b) (10pts). Suppose in fact $\beta_2 = 0$ is true and the overfitted model (B) is used. Please show that $\hat{\sigma}^2_B$ is an unbiased estimate even if the smaller model (A) is true.

   (c) (10pts). Now if model (A) is used but indeed $\beta_2 \neq 0$. Please show that $\hat{\sigma}^2_A$ is biased and compute the bias (in terms of β_1, β_2, X_1, X_2, q_1, q_2,...).

2. Consider the regression model $Y_{n\times 1} = X_{n\times q}\beta_{q\times 1} + \epsilon_{n\times 1}$, where $E(\epsilon) = 0_n$ and $Cov(\epsilon) = \sigma^2 I_n$. Denote $\hat{\beta}$ the least squares estimator of β and c is a constant vector of $\mathcal{R}^q$. The Gauss-Markov theorem states that $c'\hat{\beta}$ is the unique BLUE (best linear unbiased estimator) of $c'\beta$. Namely, $c'\hat{\beta}$ is the only estimator which has minimum variance among all linear unbiased estimators of $c'\beta$.

   (a) (18pts). Suppose X has full rank. Please prove the Gauss-Markov theorem.

   (b) (10pts) Suppose we divide the observations into two groups as $Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$ and $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$, where Y_1 and Y_2 are random vectors in $\mathcal{R}^{n_1}$ and $\mathcal{R}^{n_2}$ respectively, X_1 and X_2 are constant $n_1 \times q$ and $n_2 \times q$ matrix respectively, and $n = n_1 + n_2$. Denote $\hat{\beta}$ and $\hat{\hat{\beta}}$ the best linear unbiased estimate (BLUE) of β based on observations (X_1, Y_1) and (X, Y) respectively. Please show that $\hat{\hat{\beta}}$ has smaller mse than $\hat{\beta}$ in the sense that $E(||\hat{\hat{\beta}} - \beta||^2) \le E(||\hat{\beta} - \beta||^2)$.

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3. Let Y be a random vector in $\mathcal{R}^n$. Suppose the following model is performed,

$$Y = X_1\beta_1 + X_2\beta_2 + \epsilon$$

where X_1 and X_2 are constant $n \times q_1$ and $n \times q_2$ matrices. Suppose $X = (X_1, X_2)$ has full column rank (then so are X_1 and X_2).

(a) (18pts). Denote $\hat{Y}_1$ and $\hat{Y}$ the projection of Y on the column space of X_1 and X respectively. Please show that

$$\hat{Y} = \hat{Y}_1 + X_2^{\perp}(X_2^{\perp\prime}X_2^{\perp})^{-1}X_2^{\perp\prime}(Y - \hat{Y}_1)$$

where $X_2^{\perp} = (I - P_{X_1})X_2$ and $P_{X_1} = X_1(X_1'X_1)^{-1}X_1'$ the projection matrix onto the column space of X_1. (Hint. If $V \perp W$, then $p(y|V+W) = p(y|V) + p(y|W)$.)

(b) (18pts). Assume the column space of X_1 contains 1_n and $q_2 = 1$. The t-statistic for testing $H_0 : \beta_2 = 0$ is $t = \frac{\hat{\beta}_2}{\sqrt{S^2/||X_2^{\perp}||^2}}$ where $S^2 = \frac{||Y - \hat{Y}^2||^2}{n-q_1-1}$. Let R_1^2 and R^2 be the squared multiple correlation coefficient (coefficient of determination, the proportion of the total sum of squares due to regression) of Y with X_1 and X respectively. Please show that

$$R_1^2 = (1 + \frac{t^2}{n-q_1-1})R^2 - \frac{t^2}{n-q_1-1}.$$

(Hint. $t^2S^2 = \hat{\beta}_2^2||X_2^{\perp}||^2 = ||\hat{Y} - \hat{Y}_1||^2$.)

考試科目	線性模式專題	卷別	第二卷	考試日期	102年9月9日 星期一

1(12%)(Gauss-Markov Theorem) Let $y=\sum_{j=1}^{k}\beta_j x_j+\varepsilon$, where $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ denote linearly independent vectors, and $E[\varepsilon]=\mathbf{0}$, $D[\varepsilon]=\sigma^2 I$. Let $\eta=\sum_{j=1}^{k} c_j \beta_j$, and η^* be a linear unbiased estimator of η, then $\text{var}(\eta^*) \geq \text{var}(\hat{\eta})$, where $\hat{\eta}=\sum_{j=1}^{k} c_j \hat{\beta}_j$, and $\hat{\beta}_j$ denotes the regular least square estimator of β_j, $j=1, 2, \ldots, k$.

2(10%) Let $\mathbf{x}_1=\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$, $\mathbf{x}_2=\begin{bmatrix}1\\-1\\1\\0\end{bmatrix}$, $\mathbf{x}_3=\begin{bmatrix}0\\1\\1\\-1\end{bmatrix}$, $V_1=L(\mathbf{x}_1)$, $V_2=L(\mathbf{x}_2, \mathbf{x}_3)$, and $V=V_1 \oplus V_2$. Obtain the subspaces V_1, V_2, V, $V_1^{\perp}$, $V_2^{\perp}$, and $V^{\perp}$.

3(12%) Let Ω be the vector space of all 2x3 matrices, and $\mathbf{y}=\begin{bmatrix}20 & 11 & 8\\10 & 5 & 12\end{bmatrix}$. Find a set of orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_6\}$ such that $v_1=\begin{bmatrix}1 & 1 & 1\\1 & 1 & 1\end{bmatrix}$ and $v_2=\begin{bmatrix}1 & 1 & 1\\-1 & -1 & -1\end{bmatrix}$. And suppose that $V_0=L(\mathbf{v}_1)$, $V_R=L(\mathbf{v}_2)$, $V_C=L(\mathbf{v}_3, \mathbf{v}_4)$, and $V^{\perp}=L(\mathbf{v}_5, \mathbf{v}_6)$.

a(6%) Assume that $\hat{y}_i=p(y|V_i)$, $i=0, R, C$. Compute $\hat{y}_0$, $\hat{y}_R$, $\hat{y}_C$, $\hat{y}=\hat{y}_0+\hat{y}_R+\hat{y}_C$, and the residual $y-\hat{y}$.

b(6%) Compute the squared lengths of the vectors in a. Verify that the Pythagorean equation applies.

4(12%) Let $X_1, X_2, \ldots, X_{n_1}$ and $Y_1, Y_2, \ldots, Y_{n_2}$ be random samples taken from normal populations $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$, respectively. Specify the sampling distributions of the following statistics, as well as the constant K.

a(4%) $K[\Sigma(X_i-\mu_0)^2/\Sigma(Y_i-\bar{Y})^2]$.

b(4%) $K\dfrac{(X_1-a)^2+(X_2-a)^2}{S_2^2}$.

c(4%) $K\dfrac{X_1+X_2}{|Y_1-Y_2|}$.

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5(16%) Consider the model for one-way analysis of variance. The recorded responses are in the following table.

Treatment 1	Treatment 2	Treatment 3	Treatment 4
3	12	8	8
5	8	10	10
7			12

a(4%) Perform the $\alpha=0.05$ level F-test for $H_0: \mu_1 = \mu_2 = \mu_3 = \mu_4$.

b(6%) Use the Scheffe method to find the 95% simultaneous confidence intervals for all the six pairwise comparisons of the form $\mu_i - \mu_j$.

c(6%) Do the same as in Part b with the Bonferoni method.

(Note: Two tables of quantiles are attached.)

6(16%) A chemist wishes to determine the percentages of impurities β_1 and β_2 in two 100g containers (1 and 2) of potassium chloride. The process she uses is able to measure the weight in grams of impurities in any 2g sample of the solution with mean equal to the true weight of the impurities and standard deviation 0.006g. She makes three measurements. Measurement #1 is on a 2g sample from container 1. Measurement #2 is on a 2g sample from container 2. Measurement #3 is on a mixture of a 1g sample from container 1 and a 1g sample from container 2.

a(6%) Develop the linear model. Express the unbiased estimators $\hat{\beta}_1$ and $\hat{\beta}_2$ for β_1 and β_2 in terms of the observations.

b(6%) Compute the covariance matrix of $\hat{\beta}_1$ and $\hat{\beta}_2$.

c(4%) Assume that $y = [0.046, 0.026, 0.052]'$, compute $\hat{\beta}_1$, $\hat{\beta}_2$.

7(22%)The following is the table of totals of a two-factor experiment. There are 2 levels of factor A, 3 levels of factor B, and 2 observations in each cell. It is given that $\Sigma\Sigma\Sigma y_{ijk}^2 = 2{,}310$.

		Factor B			
		B1	B2	B3	
Factor A	A1	22	38	30	90
	A2	14	32	20	66
		36	70	50	156

State an appropriate model. Use A_i, B_j, and $(AB)_{ij}$ to denote the row, column, and interaction indicator vectors, respectively.

考試科目	線性模式專題	卷別	第二卷	考試日期	102年9月9日 星期一

a(12%) Compute the projections $\hat{y}_0, \hat{y}_A, \hat{y}_B, \hat{y}_{AB}$, and $\hat{y}$.

b(10%) Complete the ANOVA table. Perform the appropriate F-tests and state the conclusions.

(Note: Use the matrix expression: $y = \begin{bmatrix} y_{111} & y_{121} & y_{131} \\ y_{112} & y_{122} & y_{132} \\ y_{211} & y_{221} & y_{231} \\ y_{212} & y_{222} & y_{232} \end{bmatrix}$.)

Student t-table for Bonferroni: alpha=0.05, t(1-alpha/(2*m),df)

df	m=1	m=2	m=3	m=4	m=5	m=6	m=7	m=8
1	12.706	25.452	38.188	50.923	63.657	76.390	89.123	101.856
2	4.303	6.205	7.649	8.860	9.925	10.886	11.769	12.590
3	3.182	4.177	4.857	5.392	5.841	6.232	6.580	6.895
4	2.776	3.495	3.961	4.315	4.604	4.851	5.068	5.261
5	2.571	3.163	3.534	3.810	4.032	4.219	4.382	4.526
6	2.447	2.969	3.287	3.521	3.707	3.863	3.997	4.115
7	2.365	2.841	3.128	3.335	3.499	3.636	3.753	3.855
8	2.306	2.752	3.016	3.206	3.355	3.479	3.584	3.677
9	2.262	2.685	2.933	3.111	3.250	3.364	3.462	3.547
10	2.228	2.634	2.870	3.038	3.169	3.277	3.368	3.448
11	2.201	2.593	2.820	2.981	3.106	3.208	3.295	3.370
12	2.179	2.560	2.779	2.934	3.055	3.153	3.236	3.308
13	2.160	2.533	2.746	2.896	3.012	3.107	3.187	3.256
14	2.145	2.510	2.718	2.864	2.977	3.069	3.146	3.214

Table of percentiles of F(df1,df2) distribution: $\alpha=0.05$.

df2	df1=1	2	3	4	5	6	7	8	9
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80

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1. (30%) Suppose that $Y_{1,1}, \ldots, Y_{1,n_1}, Y_{2,1}, \ldots, Y_{2,n_2}$ are $(n_1 + n_2)$ independent observations. Suppose that for $i \in \{1, 2\}$ and $1 \le j \le n_i$, each $Y_{i,j}$ is normally distributed with mean $\beta_{i,0} + \beta_{i,1}x_{i,j}$ and variance σ^2, where $x_{i,j}$ is given and $\beta_{i,0}$, $\beta_{i,1}$ and σ^2 are unknown. Suppose that for $i \in \{1, 2\}$, $x_{i,1}, \ldots, x_{i,n_i}$ are not all the same.

   (a) Find a level α F test for testing H_0: $\beta_{1,1} = \beta_{2,1}$.

   (b) Construct a $(1-\alpha)100\%$ confidence interval for $\beta_{2,1} - \beta_{1,1}$.

2. (10%) For two vectors $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in R^n, let $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$ and $\|x\| = \sqrt{\langle x, x\rangle}$. That is, $\|\cdot\|$ is the Euclidean norm on R^n and $\langle\cdot,\cdot\rangle$ is the inner product corresponding to $\|\cdot\|$. Suppose that V is a subspace of R^n. For $x \in R^n$, let $p(x|V)$ denote the projection of x on V. Show that for $u, w \in R^n$,

$$\|u - p(u|V) - p(w|V)\|^2 \le \|u\|^2 + \|w\|^2.$$

3. (30%) Suppose that $Y_1, \ldots, Y_n$ are independent observations that are normally distributed with unknown variance σ^2 and for $i = 1, \ldots, n$, $E(Y_i) = \beta_0 + \beta_1 x_{1,i} + \beta_2 x_{2,i}$, where $x_{1,i}$'s and $x_{2,i}$'s are given and β_0, β_1 and β_2 are unknown parameters. Suppose that $x_{2,i} = 1 + 0.5x_{1,i}$ for $i = 1, \ldots, n$ and $x_{1,1}, \ldots, x_{1,n}$ are not all the same.

   (a) Is β_1 estimable? If so, give a $(1-\alpha)100\%$ confidence interval for β_1. If not, explain why.

   (b) Is $\beta_0 + \beta_2$ estimable? If so, give a $(1-\alpha)100\%$ confidence interval for $\beta_0 + \beta_2$. If not, explain why.

4. (30%) Suppose that $(Y_{1,1}, Y_{2,1}), \ldots, (Y_{1,2n}, Y_{2,2n})$ are independent random vectors such that for each $i \in \{1, \ldots, 2n\}$, $(Y_{1,i}, Y_{2,i})$ follows a bivariate normal distribution. Suppose that for $k \in \{1, 2\}$ and $1 \le i \le 2n$, $E(Y_{k,i}) = \mu_k$ and $Var(Y_{1,i}) = \sigma^2 = Var(Y_{2,i})$, where μ_1, μ_2 and σ^2 are unknown parameters. Suppose that

$$Cov(Y_{1,i}, Y_{2,i}) = \begin{cases} 0.5\sigma^2 & \text{if } 1 \le i \le n; \\ \sigma^2 & \text{if } (n+1) \le i \le 2n. \end{cases}$$

   (a) Find the maximum likelihood estimator for $\mu_1 - \mu_2$.

   (b) Construct a level α F test for testing $H_0 : \mu_1 = \mu_2$.