國立政治大學 101 學年度博士班資格考考試命題紙

考試科目	線性模式專題	卷別	第一卷	考試日期	101年9月10日 星期一

1. Let $\mathbf{y} \sim N_n(\mathbf{X\beta}, \sigma^2\mathbf{I}_n)$, where $\mathbf{X}$ is $n \times p$ of rank $k < p \le n$, and let $\hat{\boldsymbol\beta} = (\mathbf{X'X})^- \mathbf{X'y}$.

   (a) If $\mathbf{C}$ is $m \times p$ of rank $m \le k$ such that $\mathbf{C\beta}$ is a set of m linearly independent estimable functions, show that

   (i) $\mathbf{C}(\mathbf{X'X})^-\mathbf{C}'$ is nonsingular and invariant to $(\mathbf{X'X})^-$;

   (ii) $\mathbf{C}\hat{\boldsymbol\beta} \sim N_m(\mathbf{C\beta}, \sigma^2\mathbf{C}(\mathbf{X'X})^-\mathbf{C}')$;

   (iii) $SSH/\sigma^2 = (\mathbf{C}\hat{\boldsymbol\beta})'\left[\mathbf{C}(\mathbf{X'X})^-\mathbf{C}'\right]^{-1}\mathbf{C}\hat{\boldsymbol\beta}\big/\sigma^2 \sim \chi^2(m, \lambda)$. What is λ?

   (iv) $SSE/\sigma^2 = \mathbf{y}'\left[\mathbf{I}_n - \mathbf{X}(\mathbf{X'X})^-\mathbf{X}'\right]^{-1}\mathbf{y}\big/\sigma^2 \sim \chi^2(n-k)$;

   (v) SSH and SSE are independent.

   (b) Propose a test statistic that can be used to test $H_0 : \mathbf{C\beta} = \mathbf{0}$. What is the corresponding null sampling distribution?

2. Let $\mathbf{X} = (\mathbf{X}_1, \mathbf{X}_2)$ and $\boldsymbol\beta' = (\boldsymbol\beta_1', \boldsymbol\beta_2')$. If we fit the model $\mathbf{y} = \mathbf{X}_1\boldsymbol\beta_1^* + \boldsymbol\varepsilon^*$ when the correct model is $\mathbf{y} = \mathbf{X\beta} + \boldsymbol\varepsilon = \mathbf{X}_1\boldsymbol\beta_1 + \mathbf{X}_2\boldsymbol\beta_2 + \boldsymbol\varepsilon$ with $\mathrm{cov}(\mathbf{y}) = \sigma^2\mathbf{I}_n$. Let $\hat{\boldsymbol\beta}_1^* = (\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1'\mathbf{y}$, and $\hat{\boldsymbol\beta} = \begin{pmatrix} \hat{\boldsymbol\beta}_1 \\ \hat{\boldsymbol\beta}_2 \end{pmatrix} = (\mathbf{X'X})^{-1}\mathbf{X'y}$.

   (a) Show that $E(\hat{\boldsymbol\beta}_1^*) = \boldsymbol\beta_1 + \mathbf{A}\boldsymbol\beta_2$, where $\mathbf{A} = (\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1\mathbf{X}_2$. State a condition that $\hat{\boldsymbol\beta}_1^*$ will be an unbiased estimator of $\boldsymbol\beta_1$.

   (b) Let $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2)$ be a particular row vector of $\mathbf{X} = (\mathbf{X}_1, \mathbf{X}_2)$. Show that $E(\mathbf{x}_1\hat{\boldsymbol\beta}_1^*) \ne \mathbf{x}_1\boldsymbol\beta_1$ and $E(\mathbf{x}_1\hat{\boldsymbol\beta}_1^*) \ne \mathbf{x\beta}$.

   (c) Let s_1^2 be the estimator of σ^2 obtained using the model $\mathbf{y} = \mathbf{X}_1\boldsymbol\beta_1^* + \boldsymbol\varepsilon^*$. Find $E(s_1^2)$ and show that $E(s_1^2) \ne \sigma^2$.

考試科目	線性模式專題	卷　別	第二卷	考試日期	101年9月10日 星期一

1. (25%) If $\boldsymbol{y}$ is $MN_n(\boldsymbol{X\beta}, \sigma^2\boldsymbol{I})$, show that the likelihood ratio test for $H_0 : \boldsymbol{\beta} = \boldsymbol{0}$ can be based on

$$F = \frac{\hat{\boldsymbol{\beta}}^T \boldsymbol{X}^T \boldsymbol{y}/(k+1)}{(\boldsymbol{y}^T\boldsymbol{y} - \hat{\boldsymbol{\beta}}^T \boldsymbol{X}^T \boldsymbol{y})/(n-k-1)},$$

where $\hat{\boldsymbol{\beta}} = (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{y}$. We reject H_0 if $F > F_{\alpha,k+1,n-k-1}$.

2. (25%) Consider a group-wise heterosedastic regression model

$$y_i = \boldsymbol{x}_i^T \boldsymbol{\beta} + \epsilon_i, \quad i = 1, \cdots, n,$$

and $E(\epsilon_i|\boldsymbol{x}_i) = 0$. The n observations are grouped into G groups, each with n_g observations. The slope vector is the same in all group, but within group g:

$$Var(\epsilon_{jg}|\boldsymbol{x}_{jg}) = \sigma_g^2, \quad j = 1, \cdots, n_g, \ g = 1, \cdots, G.$$

If the variances are known, please find the generalized least squares estimator, denoted by $\hat{\boldsymbol{\beta}}$, for $\boldsymbol{\beta}$ and $Var(\hat{\boldsymbol{\beta}})$.

3. (25%)

   (1) Let $\sigma_i^2(\eta_i)$ be the variance of the random variable y_i with mean η_i. For any function $f(y)$ of y with continuous first derivative $f'(y)$ and finite second derivative $f''(y)$, please find the so-called variance stabilizing transformation, f, which would make $Var(f(y_i))$ approximately a constant.

   (2) Consider the case where y_i is a counted variable and hence $\sigma_i^2(\eta_i) \approx \eta_i$. Obtain a variance stabilizing transformation for y_i.

   (3) To conduct a study of factors affecting waiting time, one assumes that the response variable waiting time y_i has a negative exponential distribution for which $E(y_i) = \lambda_i$ and $Var(y_i) = \lambda_i^2$. Obtain a variance stabilizing transformation for y_i.

4. (25%) Please first describe the Gauss-Markov theorem and then prove it.

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1(20%) There is this two-way factorial design. The factors are both of fixed effect. Complete the ANOVA table and come up with necessary conclusions.

		Factor B		
		B1	B2	B3
Factor A	A1	$Y_{111}=6$ $Y_{112}=10$ $Y_{113}=11$	$Y_{121}=13$ $Y_{122}=15$	$Y_{131}=14$ $Y_{132}=22$
	A2	$Y_{211}=12$ $Y_{212}=15$ $Y_{213}=19$ $Y_{214}=18$	$Y_{221}=31$	$Y_{231}=18$ $Y_{232}=9$ $Y_{233}=12$

2(12%) Assume that $\mathbf{X}=\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \sim N\left(\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 6 & 3 & 3 \\ 3 & 5 & -2 \\ 3 & -2 & 9 \end{bmatrix}\right)$.

a(6%) Obtain the best linear predictor $\hat{X}_1$ of X_1 in terms of X_2 and X_3.

b(6%) Calculate $\mathrm{var}(X_1-\hat{X}_1)$.

3(16%) The effect of anesthetics on dogs are studied using a design described as follows. Ten dogs were initially given the drug pentobarbital to put to sleep. Each dog was then administered carbon dioxide (CO_2) at each of three different pressure levels (low, medium, and high). Next, halothane (H) was added, and the administration of CO_2 was repeated. The response, time in milliseconds between heartbeats of each dog, were recorded and the treatment means were shown in the table below.

	CO_2 pressure level low	med	high
Halothane present	505	503	479
Halothane absent	431	405	368

Compute the sums of squares for the main and interaction effects on the heartbeat rate of the dogs due to different pressure levels of CO_2 and presence/absence of halothane.

4(18%) Suppose $\Omega=R_4$, $\mathbf{x}_1=[1, 1, 1, 1]'$, $\mathbf{x}_2=[0, 1, 0, 1]'$, $\mathbf{x}_3=[1, 0, -1, 0]'$, $\mathbf{x}_4=[0, 3, 1, 3]'$, $V=L(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4)$ be a subspace of Ω, and $\eta=\beta_1+\beta_2+2\beta_3+2\beta_4$.

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a(6%) Verify that η is estimable. Compute the vector **a** such that $\hat{\eta}$=(**a**, **y**) is the BLUE of η.

b(8%) Give a vector **d** different from the vector **a** such that η^*=(**d**, **y**) be another linear unbiased estimator of η. Show that p(**d**|V)=**a**, and $var(\eta^*)-var(\hat{\eta})>0$.

c(4%) Now it is given that **y**=$[4, 6, 2, 8]'$, compute $\hat{\eta}$ and η^*.

<u>5(16%)</u>The joint probability mass function of the pair of random variables (X, Y) is tabulated as follows.

		X		
		0	1	2
Y	0	0.1	0.3	0.1
	1	0.2	0.2	0.1

Find the least squares predictor g(**X**)=E[Y|**X**], and the linear least squares predictor $h(\mathbf{X}) = \hat{Y} = \mu_y + \rho\sigma_Y(\mathbf{X}-\mu_X)/\sigma_X$. Show that g(**X**) and h(**X**) are unbiased estimators of E[Y]. Find var(g(**X**)), $E[(Y-g(\mathbf{X}))^2]$, var($\hat{Y}$), and $E[(Y-\hat{Y})^2]$.

<u>6(18%)</u> Let $\mathbf{y}=\begin{bmatrix} y_{11} & y_{21} & y_{31} \\ y_{12} & y_{22} & y_{32} \\ y_{13} & y_{23} & y_{33} \\ y_{14} & y_{24} & y_{34} \\ & y_{25} & \end{bmatrix}$ be the matrix of responses. Let $\mathbf{J}_1=\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ & 0 & \end{bmatrix}$, $\mathbf{J}_2=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ & 1 & \end{bmatrix}$, and $\mathbf{J}_3=\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ & 0 & \end{bmatrix}$ be the indicator matrices, and $\mathbf{x}=$

$\begin{bmatrix} x_{11} & x_{21} & x_{31} \\ x_{12} & x_{22} & x_{32} \\ x_{13} & x_{23} & x_{33} \\ x_{14} & x_{24} & x_{34} \\ & x_{25} & \end{bmatrix}$ be the matrix of the concomitant variable. The analysis of

covariance model,

$$\mathbf{y}=\sum_{j=1}^{3}\beta_j \mathbf{J}_j+\beta\mathbf{x}+\varepsilon,$$

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is to be used. And now we have $y=\begin{bmatrix} 16 & 24 & 34 \\ 19 & 20 & 29 \\ 27 & 30 & 39 \\ 18 & 25 & 26 \\ & 26 & \end{bmatrix}$, and $x=\begin{bmatrix} 3 & 4 & 5 \\ 4 & 4 & 5 \\ 7 & 6 & 8 \\ 6 & 5 & 6 \\ & 6 & \end{bmatrix}$.

a(6%) Estimate β_1, β_2, β_3, and β.

b(6%) Compute $\hat{Y}$ and S^2.

c(6%) Use $\alpha=0.05$ to test $H_0: \beta_1=\beta_2=\beta_3$ against the appropriate alternative.