國立政治大學 98 學年度博士班資格考考試命題紙

考試科目	數理統計專題	卷別	第一卷	考試日期	98 年 9 月 8 日 星期二

1. (60 pts; 10 pts for each part) Suppose that $(X_1, \ldots, X_n)$ is a random sample from the uniform distribution on $(0, \theta)$, where $\theta > 0$. Let $Pa(b, \lambda)$ denote the Pareto distribution with parameters b and λ. The probability density function for $Pa(b, \lambda)$ is given as follows:

$$f(x) = \lambda b^{\lambda} x^{-\lambda-1} I_{(b,\infty)}(x),$$

where $I_{(b,\infty)}$ denotes the indicator function on (b, ∞), and the parameters b and λ are positive. Consider the problem of estimating θ under the loss function

$$L(\theta, a) = \frac{(a-\theta)^2}{\theta^2}.$$

   (a) Show that the Bayes estimator for θ under the loss L with respect to the prior distribution $Pa(b, \lambda)$ is

$$\frac{(\lambda + n + 2)\max(b, X_{(n)})}{\lambda + n + 1},$$

   where $X_{(n)}$ denotes the n-th order statistic of $(X_1, \ldots, X_n)$.

   (b) Let $Y = \max(b, X_{(n)})$. Show that

$$E(Y) = \begin{cases} b & \text{if } b \geq \theta; \\ \frac{n\theta}{n+1} + \frac{b^{n+1}}{(n+1)\theta^n} & \text{if } b < \theta, \end{cases}$$

   and

$$E(Y^2) = \begin{cases} b^2 & \text{if } b \geq \theta; \\ \frac{n\theta^2}{n+2} + \frac{2b^{n+2}}{(n+2)\theta^n} & \text{if } b < \theta. \end{cases}$$

   (c) Find the Bayes risk for the Bayes estimator in Part (a).

   (d) Show that $(n+2)X_{(n)}/(n+1)$ is a minimax estimator of θ under the loss L.

   (e) Determine whether the Bayes estimator in Part (a) is consistent. Justify your answer.

   (f) For every $c \in (0, \infty)$, let g_c be the scale transform such that

$$g_c(X_1, \ldots, X_n) = (cX_1, \ldots, cX_n).$$

   Let $\mathcal{G} = \{g_c : c \in (0, \infty)\}$. Show that $\mathcal{G}$ is a group of transformation and determine whether the problem of estimating θ under the loss L is invariant under $\mathcal{G}$. Justify your answer.

2. (30 pts; 10 pts for each part) Suppose that $(X_1, \ldots, X_n)$ is a random sample from the exponential distribution with the following probability density function:

$$f(x) = \lambda e^{-\lambda x} I_{(0,\infty)}(x),$$

where $\lambda > 0$. Suppose that α is a constant in $(0, 1)$.

   (a) Find the UMP level α test for testing $H_0 : \lambda = 2$ versus $H_a : \lambda > 2$.

   (b) Find a UMPU level α test for testing $H_0 : \lambda = 2$ versus $H_a : \lambda \neq 2$.

   (c) Consider the UMPU test you found for Part (b). Is it also a UMP level α test for testing $H_0 : \lambda = 2$ versus $H_a : \lambda \neq 2$? Justify your answer.

3. (10 pts) Suppose that $X = (X_1, \ldots, X_n)$ is a random sample from $N(\theta, 1)$, where $\theta \in (0, \infty)$. Consider $\bar{X} = n^{-1}\sum_{i=1}^{n} X_i$ as an estimator of θ. Show that $\bar{X}$ is inadmissible under square error loss. (Hint: consider $|\bar{X}|$.)

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考試科目	數理統計專題	卷別	第二卷	考試日期	98 年 9 月 8 日 星期二

1.(10%) Let X be a nonnegative random variable with CDF F_X. Show that, if EX exists, then $EX=\int_0^\infty [1-F_X(x)]dx$.

2.(14%) Let Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables having the p.d.f. $f(x;\theta)=\frac{1}{\theta^2}\exp(-\frac{x-\theta}{\theta^2})\cdot I_{[\theta,\infty)}(x)$. Obtain a $100(1-\alpha)\%$ confidence interval for the parameter θ.

3.(24%) Let Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables having the geometric p.d.f. $f(x;\theta)=\theta(1-\theta)^x\cdot I_{\{0,1,\ldots\}}(x)$.

a(12%) Find the UMVUE of θ if it exits.

b(12%) For a squared-error loss function, find the Bayes estimator of θ with respect to a uniform prior distribution. $\theta \sim (0,1)$

4.(12%) Let $Y_1<Y_2<Y_3<Y_4<Y_5$ be the order statistics of a random sample of size $n=5$ from a distribution with p.d.f. $f(x;\theta)=\frac{1}{2}\exp(-|x-\theta|)$. Find the likelihood ratio test for testing H_0: $\theta=0$ against H_1: $\theta\neq 0$.

5.(12%) Let the random variable X have the distribution with p.d.f. $f(x;\theta)=\theta x^{\theta-1}\cdot I_{(0,1)}(x)$. For a random sample of size 2, find the most powerful size-α test of H_0: $\theta=1$ against H_1: $\theta=2$, where $\alpha=\frac{1}{2}(1-\ln 2)$.

6.(16%) Let $X_1, X_2, \ldots, X_m$ and $Y_1, Y_2, \ldots, Y_n$ be independent samples from two normal distributions $N(\mu_1,\sigma^2)$ and $N(\mu_2,\sigma^2)$, respectively, where σ^2 is the common but unknown variance.

a(8%) Find the size-α likelihood ratio test for testing H_0: $\mu_1=\mu_2=0$ against the general alternatives. Bring in the F distribution.

b(8%) Find the size-α likelihood ratio test for testing $H_0: \mu_1\leq\mu_2$ versus $H_1: \mu_1>\mu_2$. Show specifically how Student-t distribution works here.

7.(12%) State and prove the Neyman-Pearson lemma.

考試科目	數理統計專題	卷　別	第三卷	考試日期	98年9月8日 星期二

1.(5%) Let $X_1, X_2, X_3, \ldots$ be random variables. Find an example for the following case: $X_n \to_p X$, but $\{X_n\}$ does not converge to X a.s..

2.(14%) (a) (7%) If X and Y are independent random variables and $E|X|^a < \infty$ for some $a \geq 1$ and EY=0, then $E|X+Y|^a \geq E|X|^a$.

(b)(7%) If X and Y are independent random variables and $E|X|^a < \infty$ for some $a \geq 1$ and $E|Y| < \infty$, then $E|X+Y|^a \geq E|X+EY|^a$.

3.(6%) Let X_1, X_2, X_3 be a random sample of size 3 from the Bernoulli distribution. Show that the statistic

$$S_1 = u(X_1, X_2, X_3) = X_1 + X_2 + X_3$$

is a sufficient statistic but that the statistic

$$S_2 = v(X_1, X_2, X_3) = X_1 X_2 + X_3$$

is not a sufficient statistic.

4.(25%) Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables having a uniform distribution $(0, \theta)$ where $\theta > 0$ is unknown and let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be the corresponding order statistics.

(a) (5%) Show that $X_{(n)}$ is a complete sufficient statistic.

(b) (5%) Is this family a member of the exponential family? Justify your answer.

(c) (5%) Find k such that $E(kX_{(1)}) = \theta$ and compute $E(kX_{(1)})|X_{(n)})$.

(d) (5%) Show that $\frac{X_{(1)}}{X_{(n)}}$ is an ancillary statistic and compute $E\left(\frac{X_{(1)}}{X_{(n)}}\right)$.

(e)(5%) Consider to test that the null and alternative hypotheses are $H_0 : \theta = \theta_0$ vs. $H_1 : \theta \neq \theta_0$ where θ_0 is a known constant. Describe your test statistic and justify your answer.

5.(20%) Consider the bivariate discrete distribution with probability mass function

$$p(x, y; \alpha, \beta, \gamma) = c\frac{\alpha^x \beta^y \gamma^{xy}}{x!y!}, \ x=0, 1, 2, \ldots, y=0, 1, 2, \ldots, \text{ where } \alpha \geq 0,\ \beta \geq 0,$$

$0 \leq \gamma \leq 1$, and c is a normalizing constant. Consider a random sample of size n

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drawn from the bivariate distribution $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$. We want to construct a UMP test of size α of the null hypothesis that X and Y are independent against the alternative that X and Y are dependent.

(a) (5%) Determine the conditional distribution of Y given X, show that it's Poisson.

(b) (3%) Give the hypotheses of interest in terms of the parameters.

(c) (6%) Give the minimal sufficient statistics.

(d) (6%) Conduct the test with a sample of n=2, $(x_1, y_1)=(1, 1)$, and $(x_2, y_2)=(3, 2)$.

7.(11%) Let X be a single observation from the discrete p.d.f.

$f_\theta(x)=[x!(1-e^{-\theta})]^{-1}\theta^x e^{-\theta} I_{\{1,2,\ldots\}}(x)$, where $\theta > 0$ is unknown. Consider the estimation of $\theta/(1-e^{-\theta})$ under the squared error loss.

(a) (3%) Show that the estimator X is admissible.

(b) (4%) Show that X is not minimax unless $\sup_\theta R_T(\theta)=\infty$ for any estimator T=T(X).

(c) (4%) Find a loss function under which X is minimax and admissible.

8.(19%) Assume that the random variables, Y~N(0, 1), X~$N(\mu, \sigma^2)$, where $\mu \in R$ and $\sigma^2 > 0$ are unknown parameters, are independent. Now we are interested in the probability $P_{\mu,\sigma}(X > Y)$, that is, we want to estimate and to test hypotheses concerning this quantity on the basis of a random sample of n observations on X.

(a) (4%) Show that $E_{\mu,\sigma}\Phi(X) = P_{\mu,\sigma}(X > Y)$, where $\Phi(\,)$ denotes the CDF of N(0, 1).

(b) (3%) Show that $P_{\mu,\sigma}(X > Y) = \Phi(\frac{\mu}{\sqrt{1+\sigma^2}})$.

(c) (4%) Obtain the UMVUE of $P_{\mu,\sigma}(X > Y)$ by using a conditional expectation of the form

$E_{\mu,\sigma}[g(X_1) \mid \tau_1(X_1,X_2,\ldots,X_n) = t_1, \tau_2(X_1,X_2,\ldots,X_n) = t_2]$, where τ_1 and τ_2 are the classical minimal sufficient statistics.

(d) (8%) Now derive the UMP size α test for the following hypotheses:

$$H_0 : P_{\mu,\sigma}(X > Y) \le \frac{1}{2} \quad \text{versus} \quad H_1 : P_{\mu,\sigma}(X > Y) > \frac{1}{2}.$$