國立政治大學 98 學年度博士班資格考考試命題紙

考試科目	數理統計專題	卷別	第一卷	考試日期	99 年 2 月 8 日 星期一

1. (20 points)
   (a) (10 points) Let X be a nonnegative random variable. Show that
   $$EX = \int_0^\infty P(X > t)dt.$$
   (b) (10 points) If X only takes nonnegative integer values, show that
   $$EX = \sum_{t=1}^\infty P(X \geq t) = \sum_{t=0}^\infty P(X > t).$$

2. (20 points)
   Let $X_1, ..., X_n$ be normally distributed random variables satisfying the following model
   $$X_i = \theta X_{i-1} + e_i, \quad i = 1, ..., n,$$
   where $X_0 = 0$ and $e_1, ..., e_n$ are independent $N(0, \sigma^2)$ random variables.
   (a) (10 points) Find the joint density of $X_1, ..., X_n$.
   (b) (10 points) Derive the likelihood ratio statistic of $H_0 : \theta = 0$ v.s. $H_1 : \theta \neq 0$.

3. (20 points) Let $(X_i, Y_i), i = 1, ..., n$ be iid according to the bivariate normal distribution with parameter $(\mu_x, \mu_y, \sigma_x, \sigma_y, \rho)$. Let R denote the sample correlation coefficient, $\frac{\sum(X_i-\bar{X})(Y_i-\bar{Y})}{\sqrt{\sum(X_i-\bar{X})^2\sum(Y_i-\bar{Y})^2}}$. Consider the hypothesis testing problem $H_0 : \rho = 0$ v.s. $H_1 : \rho \neq 0$.
   (a) (5 points) Show that the distribution of R does not depend on $\mu_x, \mu_y, \sigma_x, \sigma_y$, but only on ρ.
   (b) (15 points) Suppose that $\rho = 0$. First show that the conditional distribution of $T = \sqrt{n-2}R/\sqrt{1-R^2}$ given $x_1, ..., x_n$ has the t-distribution with $n-2$ degrees of freedom provided $\sum(x_i - \bar{x})^2 > 0$. Then argue that this is therefore also the unconditional distribution of T. [Hint: Let $v_i = (x_i - \bar{x})/\sqrt{\sum(x_j - \bar{x})^2}$ so that $\sum v_i = 0, \sum v_i^2 = 1$. Then, rewrite T in terms of v and Y and make use of an orthogonal transformation from $(Y_1, ..., Y_n)$ to $(Z_1, ..., Z_n)$ such that $Z_1 = \sqrt{n}\bar{Y}, Z_2 = \sum v_i Y_i$.]

4. (15 points) The Kullback-Leibler information number is a measure of the difference between two probability distributions P and Q. It is defined as
   $$D_{KL}(P\|Q) = \int_{-\infty}^\infty p(x)\log\frac{p(x)}{q(x)}dx,$$
   where p and q are the densities of P and Q. Show that $D_{KL}(P\|Q) \geq 0$.

5. (25 points) Suppose that $X_t = \mu + \sum_{j=-\infty}^\infty \psi_j Z_{t-j}$ where $\sum_{j=-\infty}^\infty |\psi_j| < \infty$ and Z_j are iid r.v.'s with mean 0 and variance σ^2. Let $\bar{X}_n = (1/n)\sum_{t=1}^n X_t$ and $\gamma(h) = E(X_t - \mu)(X_{t+h} - \mu)$, $-\infty < h < \infty$.
   (a) (10 points) Show that $\sum_{h=-\infty}^\infty |\gamma(h)| < \infty$.
   (b) (5 points) Let $\bar{X}_n = \sum_{t=1}^n X_t$. Show that $E\left[(\bar{X}_n - \mu)^2\right] = \sum_{i=1}^n \sum_{j=1}^n \gamma(i-j)$.
   (c) (5 points) Show that $\mathrm{Var}(\bar{X}_n) \to 0$.
   (d) (5 points) Show that $n\mathrm{Var}(\bar{X}_n) \to \sum_{h=-\infty}^\infty \gamma(h)$.

考試科目	數理統計專題	卷別	第二卷	考試日期	99 年 2 月 8 日 星期一

1.(14%) Let $X_1, X_2, X_3, \ldots$ be a sequence of random variables. Find examples for the following cases and support your answers.

a(7%) $X_n \to_d X$, but $\{X_n\}$ does not converge to X in probability.

b(7%) $X_n \to_p X$, but $\{g(X_n)\}$ does not converge to g(X) in probability for some function g.

2.(12%) Let F_1 and F_2 be two c.d.f.'s. Show that $F_1(x) \le F_2(x)$ for all x if and only if $\int g(x)dF_2(x) \le \int g(x)dF_1(x)$ for any nondecreasing function g.

3.(12%) Let $X_1, X_2, \ldots, X_n$ be a random sample from the p.d.f. $\frac{1+\theta x}{2} I_{(-1,1)}(x)$, where $\theta \in (-1, 1)$ is an unknown parameter. Find a consistent estimator of θ. Is your estimator $\sqrt{n}$-consistent? (While a_n is a sequence of positive numbers diverging to ∞, the estimator $T_n(X)$ is said to be a_n-consistent of θ if and only if $a_n |T_n(X) - \theta| = O_P(1)$ w.r.t. any probability measure P.)

4.(12%) Let $Y_1 < Y_2 < Y_3 < Y_4 < Y_5$ be the order statistics of a random sample of size n = 5 from a distribution with p.d.f. $f(x;\theta) = \frac{1}{2}\exp(-|x-\theta|)$. Find the likelihood ratio test for testing H_0: $\theta = 0$ against H_1: $\theta \ne 0$.

5.(16%) Let X be a single observation from the double exponential distribution DE(μ, θ) with $\mu = 0$ and $\theta > 0$. Find the UMVUE's of both θ and $\frac{1}{(1+\theta)}$. And, in each case, determine whether the variance of the estimator attains the Cramer-Rao lower bound. (The double exponential distribution DE(μ, θ)has the p.d.f. $\frac{1}{2\theta} e^{-|x-\mu|/\theta}$.)

6.(12%) Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with p.d.f. $\frac{1}{\theta} e^{-(x-b)/\theta}$, where $b \in R$ and $\theta > 0$. Find a size α UMP test for testing H_0: $b = b_0$, $\theta = \theta_0$ against H_1: $b < b_0$, $\theta < \theta_0$.

7.(10%) Let $Y_1 < Y_2 < \ldots < Y_n$ be the order statistics of a random sample of size n from the uniform distribution U[0, θ]. Show that a shortest $100(1-\alpha)$% confidence interval of θ of the form $[a Y_n, b Y_n]$ where a and b are positive numbers, can be achieved..

考試科目	數理統計專題	卷別	第二卷	考試日期	99年2月8日 星期一

8.(12%) Let $X_1, X_2, \ldots, X_m$ and $Y_1, Y_2, \ldots, Y_n$ be independent random samples from two exponential distributions $\frac{1}{\theta_1}e^{-x/\theta_1}$ and $\frac{1}{\theta_2}e^{-y/\theta_2}$, respectively, where θ_1 and θ_2 are positive but unknown parameters. Find the size-α likelihood ratio test for testing $H_0: \theta_1 \le \theta_2$ versus $H_1: \theta_1 > \theta_2$.

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考試科目	數理統計專題	卷別	第三卷	考試日期	99 年 2 月 8 日 星期一

1. (60 pts; 10 pts for each part) Suppose that $(X_1, \ldots, X_n)$ is a random sample from the uniform distribution on $(0, \theta)$, where $\theta > 0$. Let $Pa(b, \lambda)$ denote the Pareto distribution with parameters b and λ. The probability density function for $Pa(b, \lambda)$ is given as follows:

$$f(x) = \lambda b^{\lambda} x^{-\lambda-1} I_{(b,\infty)}(x),$$

where $I_{(b,\infty)}$ denotes the indicator function on (b, ∞), and the parameters b and λ are positive. Consider the problem of estimating θ under the loss function

$$L(\theta, a) = \frac{(a-\theta)^2}{\theta^2}.$$

   (a) Show that the Bayes estimator for θ under the loss L with respect to the prior distribution $Pa(b, \lambda)$ is

$$\frac{(\lambda + n + 2)\max(b, X_{(n)})}{\lambda + n + 1},$$

   where $X_{(n)}$ denotes the n-th order statistic of $(X_1, \ldots, X_n)$.

   (b) Let $Y = \max(b, X_{(n)})$. Show that

$$E(Y) = \begin{cases} b & \text{if } b \geq \theta; \\ \frac{n\theta}{n+1} + \frac{b^{n+1}}{(n+1)\theta^n} & \text{if } b < \theta, \end{cases}$$

   and

$$E(Y^2) = \begin{cases} b^2 & \text{if } b \geq \theta; \\ \frac{n\theta^2}{n+2} + \frac{2b^{n+2}}{(n+2)\theta^n} & \text{if } b < \theta. \end{cases}$$

   (c) Find the Bayes risk for the Bayes estimator in Part (a).

   (d) Show that $(n+2)X_{(n)}/(n+1)$ is a minimax estimator of θ under the loss L.

   (e) Determine whether the Bayes estimator in Part (a) is consistent. Justify your answer.

   (f) For every $c \in (0, \infty)$, let g_c be the scale transform such that

$$g_c(X_1, \ldots, X_n) = (cX_1, \ldots, cX_n).$$

   Let $\mathcal{G} = \{g_c : c \in (0, \infty)\}$. Show that $\mathcal{G}$ is a group of transformation and determine whether the problem of estimating θ under the loss L is invariant under $\mathcal{G}$. Justify your answer.

2. (30 pts; 10 pts for each part) Suppose that $(X_1, \ldots, X_n)$ is a random sample from the exponential distribution with the following probability density function:

$$f(x) = \lambda e^{-\lambda x} I_{(0,\infty)}(x),$$

where $\lambda > 0$. Suppose that α is a constant in $(0, 1)$.

   (a) Find the UMP level α test for testing $H_0 : \lambda = 2$ versus $H_a : \lambda > 2$.

   (b) Find a UMPU level α test for testing $H_0 : \lambda = 2$ versus $H_a : \lambda \neq 2$.

   (c) Consider the UMPU test you found for Part (b). Is it also a UMP level α test for testing $H_0 : \lambda = 2$ versus $H_a : \lambda \neq 2$? Justify your answer.

3. (10 pts) Suppose that $X = (X_1, \ldots, X_n)$ is a random sample from $N(\theta, 1)$, where $\theta \in (0, \infty)$. Consider $\bar{X} = n^{-1}\sum_{i=1}^{n} X_i$ as an estimator of θ. Show that $\bar{X}$ is inadmissible under square error loss. (Hint: consider $|\bar{X}|$.)