(第一卷)

1. (40%) Given the linear model $\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{\epsilon}$, where $\boldsymbol{X}$ is $n \times p$ of rank p and $\epsilon \overset{iid}{\sim} MN(\boldsymbol{0}, \sigma^2 \boldsymbol{I}_n)$, we wish to test

$$H_0 : \boldsymbol{A\beta} = \boldsymbol{c}, \tag{1}$$

where $\boldsymbol{A}$ is $q \times p$ of rank q.

(a) Please derive a likelihood ratio test for (1). (10%)

(b) Please derive an F-test for (1). (25%)

(c) Compare the results of parts (a) and (b). (5%)

2. (15%) Suppose that $Y_1, \cdots, Y_n$ are gamma random variables with density functions

$$f(y_i) = \frac{1}{\Gamma(\alpha)\lambda_i^\alpha} y_i^{\alpha-1} \exp(-y_i/\lambda_i).$$

Find a transformation that will make the variances of the Y_i's approximately equal.

3. (25%) Suppose that the true model is $\boldsymbol{\theta} = \boldsymbol{X\beta} + \boldsymbol{W}\gamma$, but the postulated model is $\boldsymbol{\theta} = \boldsymbol{X\beta}$. What would this have any effects upon the estimates of parameters and the expectation of the residual mean square value (i.e. MSE)?

4. (20%) Let $Y_{ijk} = \mu_{ij} + \epsilon_{ijk}$, where

$$\begin{aligned} \mu_{ij} &= \bar{\mu}_{..} + (\bar{\mu}_{i.} - \bar{\mu}_{..}) + (\bar{\mu}_{ij} - \bar{\mu}_{i.}) \\ &= \mu + \alpha_i + \beta_{ij}, \end{aligned}$$

where $\epsilon_{ijk} \overset{iid}{\sim} N(0, \sigma^2)$, $i = 1, \cdots, I$, $j = 1, \cdots, J$, and $k = 1, \cdots, K$. Please obtain the test statistics for testing hypotheses $H_0 : \beta_{ij} = 0$ for all i, j.

Linear Models (Qualifying Exam) (第二卷)

2/16/2009

30% 1. Suppose that $\mathbf{Y} \sim N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. Let

$$\mathbf{Y} = \begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{pmatrix}, \quad \boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{pmatrix}, \quad \boldsymbol{\Sigma} = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix},$$

Where $\mathbf{Y}_1$ and $\boldsymbol{\mu}_1$ are $p \times 1$, and $\boldsymbol{\Sigma}_{11}$ $p \times p$.

(a) Show that $\mathbf{Y}_1 \sim N_p(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_{11})$, $\mathbf{Y}_2 \sim N_{n-p}(\boldsymbol{\mu}_2, \boldsymbol{\Sigma}_{22})$.

(b) Show that $\mathbf{Y}_1$ and $\mathbf{Y}_2$ are independent if and only if $\boldsymbol{\Sigma}_{12} = \mathbf{0}$.

(c) If $\boldsymbol{\Sigma}_{22} > 0$, find the conditional distribution of $\mathbf{Y}_1$ given $\mathbf{Y}_2$.

20% 2. Suppose $\mathbf{X}$ is an $n \times (k+1)$ matrix of rank $k+1 < n$, and its first column is $\mathbf{j}$.

Show that $1/n \le h_{ii} \le 1, \ \forall i = 1, \ldots, n$ and $-0.5 \le h_{ij} \le 0.5, \ \forall i \ne j$.

50% 3. Consider the model $y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_k + \varepsilon_{ijk}$, $i = 1,2,\ j = 1,2,\ k = 1,2$.

(a) Write $\mathbf{X}'\mathbf{X}$, $\mathbf{X}'\mathbf{y}$, and the normal equations.

(b) Find a set of linearly independent estimable functions.

(c) Define appropriate side conditions and find the resulting solution to the normal equations.

(d) Show that $H_0: \alpha_1 = \alpha_2$ is testable. Find $\hat{\boldsymbol{\beta}}'\mathbf{X}'\mathbf{y} = \mathrm{SS}(\mu, \alpha, \beta, \gamma)$ and

$\hat{\boldsymbol{\beta}}_2'\mathbf{X}_2'\mathbf{y} = \mathrm{SS}(\mu, \beta, \gamma)$.

(e) Construct an analysis of variance table for the test of $H_0: \alpha_1 = \alpha_2$.

Linear Models (2/16/2009) (第三卷)

1.(10%) Assume that $\mathbf{X}=\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \sim N(\mathbf{0}, \begin{bmatrix} 4 & 2 & -3 \\ 2 & 4 & -1 \\ -3 & -1 & 6 \end{bmatrix})$,.

a(5%) Obtain the best linear predictor $\hat{X}_1$ of X_1 in terms of X_2 and X_3.

b(5%) Calculate $\mathrm{var}(X_1 - \hat{X}_1)$.

2.(8%) Let $A=\begin{bmatrix} 7 & -2\sqrt{2} \\ -2\sqrt{2} & 5 \end{bmatrix}$. Find a matrix U such that $A=UU'$.

3.(10%) Assume that $\mathbf{X}=\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \sim N(\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 & 2 & -1 \\ 2 & 5 & -1 \\ -1 & -1 & 3 \end{bmatrix})$, and that

$Q_1(\mathbf{X})=(X_1-\bar{X})^2+(X_2-\bar{X})^2+(X_3-\bar{X})^2$, and

$Q_2(\mathbf{X})=(X_1-X_2)^2+(X_2-X_3)^2+(X_1-X_3)^2$. Compute $E[Q_1(\mathbf{X})]$ and $E[Q_2(\mathbf{X})]$.

4.(12%) Give the singular value decompositions and the Moore-Penrose inverses of the matrix X.

a. $X=\begin{bmatrix} 5 & -1 \\ -1 & 5 \\ 2 & 2 \end{bmatrix}$.

b. $X=\begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$.

5.(12%) Let $X_1, X_2, \ldots, X_{n_1}$ and $Y_1, Y_2, \ldots, Y_{n_2}$ be random samples taken from normal populations $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$, respectively. Specify the sampling distributions of the following statistics.

a(3%) $[(\bar{X}-\bar{Y})-\delta]/\sqrt{\dfrac{\sigma_1^2}{n_1}+\dfrac{\sigma_2^2}{n_2}}$, where δ denotes a constant.

b(3%) $\dfrac{1}{\sigma_1^2}\sum(X_i - C_i)^2$, where $C_1, C_2, \ldots, C_{n_1}$ denote constants.

c(3%) $K\dfrac{(X_1-a)^2+(X_2-a)^2}{S_2^2}$. Please also specify constant K.

d(3%) $K\dfrac{X_1+X_2}{|Y_1-Y_2|}$. Please also specify constant K.

6.(18%) Suppose $\Omega=R_4$, $\mathbf{x}_1=[1, 1, 1, 1]'$, $\mathbf{x}_2=[0, 1, 0, 1]'$, $\mathbf{x}_3=[1, 0, -1, 0]'$, $\mathbf{x}_4=[1, 3, 2, 3]'$, $V=L(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4)$ be a subspace of Ω, and $\eta=\beta_1-\beta_2-2\beta_3+\beta_4$.

a(6%) Verify that η is estimable. Give a vector **d** such that $\eta^*=(\mathbf{d}, \mathbf{y})$ be a linear unbiased estimator of η.

b(6%) Compute the vector **a** such that $\hat{\eta}=(\mathbf{a}, \mathbf{y})$ be the BLUE of η. Show that $p(\mathbf{d}|V)=\mathbf{a}$, and $var(\eta^*)-var(\hat{\eta})>0$.

c(6%) Now it is given that $\mathbf{y}=[5, 8, 4, 9]'$, estimate σ^2.

7.(18%) We are modeling yields of blueberry of three different conditions with several plots within each condition. Let $\mathbf{y}=\begin{bmatrix} y_{11} & y_{21} & y_{31} \\ y_{12} & y_{22} & y_{32} \\ y_{13} & y_{23} & y_{33} \\ y_{14} & y_{24} & y_{34} \\ & y_{25} & \end{bmatrix}$, in which the entry y_{ij} denotes the yield of blueberry of the ith condition on the jth plot, where $j=1, 2, \ldots, n_i$, $i=1, 2, 3$, $n_1=4$, $n_2=5$, and $n_3=4$. Furthermore, let $\mathbf{J}_1=\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ & 0 & \end{bmatrix}$, $\mathbf{J}_2=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ & 1 & \end{bmatrix}$, and $\mathbf{J}_3=\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ & 0 & \end{bmatrix}$ denote the condition effect indicator matrices, and the entry x_{ij} in the matrix $\mathbf{x}=\begin{bmatrix} x_{11} & x_{21} & x_{31} \\ x_{12} & x_{22} & x_{32} \\ x_{13} & x_{23} & x_{33} \\ x_{14} & x_{24} & x_{34} \\ & x_{25} & \end{bmatrix}$ denotes the amount of fertilization for the blueberry of ith condition on the jth plot. We are now fitting the model to the yield of blueberry.

$$y=\sum_{j=1}^{3}\beta_j \mathbf{J}_j+\beta x+\varepsilon.$$

And now we have $\mathbf{y}=\begin{bmatrix}16 & 24 & 32\\ 19 & 20 & 35\\ 24 & 29 & 39\\ 21 & 26 & 26\\ & 26 & \end{bmatrix}$, and $\mathbf{x}=\begin{bmatrix}3 & 4 & 5\\ 4 & 4 & 5\\ 7 & 6 & 6\\ 6 & 5 & 4\\ & 6 & \end{bmatrix}$.

a(6%) Estimate β_1, β_2, β_3, and β.

b(6%) Compute $\hat{y}$ and S^2.

c(6%) Use α=0.05 to test H_0: $\beta_1=\beta_2=\beta_3$ against the appropriate alternative.

8.(12%) Let M be a k×k positive definite matrix. Let A be a q×k matrix of rank q. Let C be the row space of A, and $Q=A'[AM^{-1}A']^{-1}A$. Then show that, for any $b\in R_k$, (1) $\sup_{c\in C}\frac{c'b^2}{c'M^{-1}c}=b'Qb$, with the supremum achieved for c=Qb, and (2)

If X is an n×k matrix such that $X'X=M$, $\hat{Y}=Xb$, and $\hat{Y}_1=XM^{-1}Qb$, then

$\hat{Y}_1=p(\hat{Y}|V_1)$, where V_1 is the column space of $XM^{-1}A'$, and $\|\hat{Y}\|=b'Qb$.