第三卷

Mathematical Statistics

9/17/2007

1.(6%) Let $\{X_n\}$ be a monotone decreasing sequence of non-negative random variables. Show that if $X_n \to_p 0$, then $X_n \to_{a.s.} 0$.

2.(14%) a(7%) If X and Y are independent random variables and $E|X|^a < \infty$ for some $a \geq 1$ and EY=0, then $E|X+Y|^a \geq E|X|^a$.

b(7%) If X and Y are independent random variables and $E|X|^a < \infty$ for some $a \geq 1$ and $E|Y| < \infty$, then $E|X+Y|^a \geq E|X+EY|^a$.

3.(6%) Let X_1, X_2, X_3 be a random sample of size 3 from the Bernoulli distribution. Show that the statistic

$S_1 = u(X_1, X_2, X_3) = X_1 + X_2 + X_3$

is a sufficient statistic but that the statistic

$S_2 = v(X_1, X_2, X_3) = X_1 X_2 + X_3$

is not a sufficient statistic.

4.(10%) Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables having the exponential distribution $E(a, \theta)$, $a \in R$, and $\theta > 0$. Show that the smallest order statistic $X_{(1)}$ has the exponential distribution $E(a, \theta/n)$ and that $2\sum_{i=1}^{n}(X_i - X_{(1)})/\theta$ has the chi-square distribution χ^2_{2n-2}.

5.(8%) Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables having the Lebesgue p.d.f.

$$f_\theta(x) = (2\theta)^{-1}[I_{(0,\theta)}(x) + I_{(2\theta,3\theta)}(x)].$$

Find a sufficient statistic for $\theta \in (0, \infty)$.

6.(20%) Consider the bivariate discrete distribution with probability mass function

$p(x, y; \alpha, \beta, \gamma) = c\dfrac{\alpha^x \beta^y \gamma^{xy}}{x!y!}$, x=0, 1, 2, ..., y=0, 1, 2, ..., where $\alpha \geq 0$, $\beta \geq 0$, $0 \leq \gamma \leq 1$, and c is a normalizing constant. Consider a random sample of size n drawn from the bivariate distribution $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$. We want to construct a UMP test of size α of the null hypothesis that X and Y are independent against the alternative that X and Y are dependent.

a(5%) Determine the conditional distribution of Y given X, show that it's Poisson.

b(3%) Give the hypotheses of interest in terms of the parameters.

c(6%) Give the minimal sufficient statistics.

d(6%) Conduct the test with a sample of n=2, (x_1, y_1)=(1, 1), and (x_2, y_2)=(3, 2).

7.(14%) Let X be a single observation from the discrete p.d.f.

$f_\theta(x)=[x!(1-e^{-\theta})]^{-1}\theta^x e^{-\theta} I_{\{1,2,\ldots\}}(x)$, where $\theta > 0$ is unknown. Consider the estimation of $\theta/(1-e^{-\theta})$ under the squared error loss.

a(4%) Show that the estimator X is admissible.

b(5%) Show that X is not minimax unless $\sup_\theta R_T(\theta)=\infty$ for any estimator T=T(X).

c(5%) Find a loss function under which X is minimax and admissible.

8.(22%) Assume that the random variables, Y~N(0, 1), X~N(μ,σ^2), where $\mu \in R$ and $\sigma^2 > 0$ are unknown parameters, are independent. Now we are interested in the probability $P_{\mu,\sigma}(X > Y)$, that is, we want to estimate and to test hypotheses concerning this quantity on the basis of a random sample of n observations on X.

a(5%) Show that $E_{\mu,\sigma}\Phi(X) = P_{\mu,\sigma}(X > Y)$, where $\Phi(\,)$ denotes the CDF of N(0, 1).

b(4%) Show that $P_{\mu,\sigma}(X > Y) = \Phi(\frac{\mu}{\sqrt{1+\sigma^2}})$.

c(5%) Obtain the UMVUE of $P_{\mu,\sigma}(X > Y)$ by using a conditional expectation of the form $E_{\mu,\sigma}[g(X_1) \mid \tau_1(X_1,X_2,\ldots,X_n)=t_1, \tau_2(X_1,X_2,\ldots,X_n)=t_2]$, where τ_1 and τ_2 are the classical minimal sufficient statistics.

d(8%) Now derive the UMP size α test for the following hypotheses:

$$H_0 : P_{\mu,\sigma}(X > Y) \le \frac{1}{2} \quad \text{versus} \quad H_1 : P_{\mu,\sigma}(X > Y) > \frac{1}{2}.$$