第二卷

Mathematical Statistics: Ph.D. Qualifier Exam

February 26, 2007
(100 minutes)

1. Let $X_1, X_2, X_3, \ldots$ be i.i.d. with $\mathcal{N}(\theta, 1)$ and $Y_n = \Phi((c - \bar{X}_n)/\sqrt{1-1/n})$, where Φ is the standard normal distribution function, c is a constant, and $\bar{X}_n = (X_1 + X_2 + \ldots + X_n)/n$. We are interested in the limiting behaviour of Y_n. Please answer the following questions.

   (a) (6pts). Denote $T_n = c - \bar{X}_n$. Find r_n and s_n such that $r_n(T_n - s_n)$ has a nondegenerate limiting distribution, and find the distribution.

   (b) (6pts). Denote $W_n = (c - \bar{X}_n)/\sqrt{1-1/n}$. Find u_n and v_n such that $u_n(W_n - v_n)$ has a nondegenerate limiting distribution, and find the distribution.

   (c) (6pts). Find a_n and b_n such that $a_n(Y_n - b_n)$ has a nondegenerate limiting distribution, and find the distribution.

2. Sometimes unbiased estimators exist, but none is UMVUE (uniformly minimum variance unbiased estimator). An example follows. Suppose that P_θ says that $Y_1, Y_2, Y_3, \ldots$ are i.i.d. with Ber(θ). Set

$$X = \begin{cases} 1 & \text{if } Y_1 = 1 \\ \text{number of trials until 2nd failure} & \text{otherwise} \end{cases},$$

   and suppose that we observe only X. Please answer the following questions.

   (a) (4pts). Please give the density function of X.

   (b) (4pts). Please find an unbiased estimator of θ.

   (c) (8pts). Please prove the following statement for general situation. Let $\mathcal{U}$ be the set of all unbiased estimators of 0. Namely,

$$\mathcal{U} = \{U(X) : \mathrm{E}_\theta(U(X)) = 0 \text{ for all } \theta\}.$$

   Then $\delta(X)$ is an UMVUE of θ if and only if, for any $U(X) \in \mathcal{U}$, $\mathrm{Cov}(\delta(X), U(X)) = 0$.

   (d) (8pts). Please characterize $\mathcal{U}$ for the example given above. Namely, find conditions on $U(\cdot)$ for $U \in \mathcal{U}$ for the example given above.

   (e) (8pts). Please show that there is no UMVUE of θ for the example given above.

3. Please answer the following questions.

   (a) (10pts). Let $\{P_\theta : \theta \in \Theta\}$ be a parametric family, and let $p_\theta(x)$ be the density of a member of the family with respect to Lebesgue measure. Assume that $\partial^2 \log p_\theta(X)/\partial x \partial \theta$ exists for all x and θ. Please prove that the family has increasing MLR (monotone likelihood ratio) if and only if $\partial^2 \log p_\theta(X)/\partial x \partial \theta \geq 0$ for all x and θ.

   (b) (4pts). Please give a similar statement for case of decreasing MLR. Proof is not required.

4. Suppose that $X \sim \mathcal{N}(\theta, 1)$ and let θ have an $\mathcal{N}(0, 1)$ prior. Suppose we are interested in estimating θ and the parameter space and action space are both $(-\infty, \infty)$. Let the loss function be $L(\theta, a) = 0$ if $a \geq \theta$ and $L(\theta, a) = 1$ if $a < \theta$, where a denotes the action (estimator).

   (a) (4pts). What is the posterior of θ?

   (b) (4pts). Please compute the Bayes risk (Posterior risk) $r(\delta(X)|X)$, where $\delta(X)$ is any decision rule.

   (c) (4pts). Please show that there is no Bayes rule.

   (d) (8pts). Please show that every decision rule is inadmissible.

   (e) (12pts). Please show that if the action space is $[-\infty, \infty]$, then there is a Bayes rule and that it is the only admissible rule.

   (f) (4pts). How do you comment on the loss function? Do you take L a nice loss function in this decision problem? Explain.