Linear Models 第一卷

9/12/2005

1. Suppose y is $N_n(\boldsymbol{\mu}, \sigma^2\mathbf{I})$ and suppose $\mathbf{X}$ is an $n \times p$ matrix of constants with rank $p < n$.
   (a) Show that $\mathbf{A} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$ and $\mathbf{I} - \mathbf{A} = \mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$ and are idempotent and find the rank of each.
   (b) If $\boldsymbol{\mu}$ is a linear combination of the columns of $\mathbf{X}$, that is, $\boldsymbol{\mu} = \mathbf{X}\boldsymbol{\beta}$ for some $\boldsymbol{\beta}$, find $E(\mathbf{y}'\mathbf{A}\mathbf{y})$ and $E(\mathbf{y}'(\mathbf{I}-\mathbf{A})\mathbf{y})$, where $\mathbf{A}$ is defined in (a).
   (c) Find the distributions of $\mathbf{y}'\mathbf{A}\mathbf{y}/\sigma^2$ and $\mathbf{y}'(\mathbf{I}-\mathbf{A})\mathbf{y}/\sigma^2$.
   (d) Show that $\mathbf{y}'\mathbf{A}\mathbf{y}$ and $\mathbf{y}'(\mathbf{I}-\mathbf{A})\mathbf{y}$ are independent.
   (e) Find the distribution of $\dfrac{\mathbf{y}'\mathbf{A}\mathbf{y}/p}{\mathbf{y}'(\mathbf{I}-\mathbf{A})\mathbf{y}/(n-p)}$.

2. (a) Consider the model $y_{ij} = \mu + \alpha_i + \beta_i x_{ij} + \varepsilon_{ij}$, $i = 1, 2, \ldots, k$, $j = 1, 2, \ldots, n$. Find a test statistic to see if $\beta_1 = \beta_2 = \cdots = \beta_k$ holds. What is it distributed?
   (b) For the model $y_{ij} = \mu + \alpha_i + \beta x_{ij} + \varepsilon_{ij}$, $i = 1, 2, \ldots, k$, $j = 1, 2, \ldots, n$. Find a test statistic for the test $H_0 : \alpha_1 = \alpha_2 = \cdots = \alpha_k$. What is it distributed?

第二卷

Linear models (94.9.)

1.(10%) Let Y=$\begin{bmatrix} 5 & 6 & 1 \\ 11 & 8 & 5 \end{bmatrix}$, $\mathbf{R}_1, \mathbf{R}_2, \mathbf{C}_1, \mathbf{C}_2$, **and** $\mathbf{C}_3$ denote the indicators of the 2 rows and 3 columns. For V=L($\mathbf{R}_1, \mathbf{R}_2, \mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_3$), find $\hat{Y}$, e=Y$-\hat{Y}$, $\|\hat{Y}\|^2$, and $\|\mathbf{e}\|^2$. Verify that e$\perp$V.

2.(16%) The joint probability mass function of the pair of random variables (X, Y) is tabulated as follows.

		X		
		0	**1**	**2**
Y	**0**	0.1	0.3	0.2
	1	0.2	0.1	0.1

Find the least squares predictor g(**X**)=E[**Y**|**X**], and the linear least squares predictor h(**X**)= $\hat{Y}=\mu_y+\rho\sigma_Y(\mathbf{X}-\mu_X)/\sigma_X$. Show that g(**X**) and h(**X**) are unbiased estimators of E[**Y**]. Find var(g(**X**)), E[(**Y**$-$g(**X**))^2], var($\hat{Y}$), and E[(**Y**$-\hat{Y}$)^2].

3.(12%) We now consider the model $Y=\Sigma\beta_j x_j+\varepsilon$, where $x_1, x_2, \ldots, x_k$ are linearly independent vectors, and $\varepsilon \sim N(0, \sigma^2 I)$. Define $V_{k-1}=L(x_1, x_2, \ldots, x_{k-1})$, $\hat{x}_k=p(x_k \mid V_{k-1})$, and $x_k^{\perp}=x_k-\hat{x}_k$. Show that $\operatorname{var}(\hat{\beta}_k)=\sigma^2/\|x_k^{\perp}\|^2$, and

$$\operatorname{cov}(\hat{\beta}_i, \hat{\beta}_j)=\frac{\sigma^2 \cos w_{ij}}{\| x_i^{\perp} \| \cdot \| x_j^{\perp} \|},$$

where w_{ij} denotes the angle between $x_i^{\perp}$ and $x_j^{\perp}$.

4.(10%) Determine whether the quadratic form $Q(x_1, x_2, x_3)=$ $13x_1^2+13x_2^2+10x_3^2-8x_1x_2+4x_1x_3-4x_2x_3$ is positive definite. Hint: One of the eigenvalues of the matrix corresponding to Q is 9.

5.(12%) The vector space V, defined as $V=L(x_1, x_2, \ldots, x_k)$, has dimension k. Find simple formulae for the coefficients a_j in $p(y|V)=\sum_j a_j x_j^{\perp}$, and show that $L(x_1^{\perp}, x_2^{\perp}, \ldots, x_k^{\perp})=V$.

6.(20%) In fitting the model $y=X\beta+\varepsilon$, where $\varepsilon \sim N(0, \sigma^2 I)$, it is given that $y=\begin{bmatrix}50\\40\\52\\47\\65\end{bmatrix}$,

$$X=\begin{bmatrix}1 & 1 & 1\\1 & 5 & 1\\1 & 5 & 2\\1 & 10 & 2\\1 & 20 & 3\end{bmatrix}, \quad \beta=\begin{bmatrix}\beta_0\\\beta_1\\\beta_2\end{bmatrix}, \quad \hat{\beta}=\begin{bmatrix}33.06\\-0.189\\10.718\end{bmatrix}, \quad \text{and that}$$

$$(X'X)^{-1}=\begin{bmatrix}2.307551 & 0.1565378 & -1.88398\\ & 0.02578269 & -0.20442\\ & & 1.977901\end{bmatrix}.$$

I. Perform the following two-sided tests with the significance level of $\alpha=0.05$.

(1) $H_0: \beta_1=0$. **(2)** $H_0: \beta_2=8$. **(3)** $H_0: 50\beta_1+\beta_2=0$.

II. Compute the Bonferroni simultaneous 95% confidence intervals for the following groups of parameters.

(1) β_0, β_1, and β_2. **(2)** $\beta_0-2\beta_2$, and $40\beta_1+\beta_2$.

7.(20%) The following is the table of totals of a two-factor experiment. There are 2 levels of factor A, 3 levels of factor B, and 4 observations in each cell. It is given that $\Sigma\Sigma\Sigma y_{ijk}^2=17579$.

		Factor B			
		B1	B2	B3	
Factor A	A1	86	102	124	312
	A2	96	122	112	330
		182	224	236	642

State an appropriate model. Use A_i, B_j, and $(AB)_{ij}$ to denote the row, column, and interaction indicator vectors, respectively. Compute the projections $\hat{Y}_0$, $\hat{Y}_A$, $\hat{Y}_B$, $\hat{Y}_{AB}$, and $\hat{Y}$. Complete the ANOVA table. Perform the appropriate F-tests and state the conclusions.

Linear Models 第三卷

9/12/2005

1. Let $\boldsymbol{x}_1 = (1,1,1,1,1,1)'$, $\boldsymbol{x}_2 = (3,-1,4,6,3,3)'$, $\boldsymbol{x}_3 = (7,3,2,0,3,3)'$, $\boldsymbol{x}_4 = (8,4,9,-5,4,4)'$, $\boldsymbol{Y} = (4,36,44,12,16,8)'$, $V = \mathcal{L}(\boldsymbol{x}_1, \boldsymbol{x}_3, \boldsymbol{x}_3, \boldsymbol{x}_4)$. Suppose we wish to test $H_0 : \beta_4 = 0, \beta_2 = \beta_3$.

(1) Find two matrices $\boldsymbol{A}$ so that $H_0 \Leftrightarrow \boldsymbol{A\beta} = 0$.

(2) Find $\hat{\boldsymbol{\beta}}$, $\hat{\boldsymbol{Y}} = \boldsymbol{X}\hat{\boldsymbol{\beta}}$, and $\boldsymbol{Z} = \boldsymbol{A}\hat{\boldsymbol{\beta}}$, for one of your choices for $\boldsymbol{A}$.

(3) Define V_0 so that $H_0 \Leftrightarrow \boldsymbol{\theta} \in V_0$ and find $\hat{\boldsymbol{Y}}_0 = p(\boldsymbol{Y}|V_0)$, $\boldsymbol{Y} - \hat{\boldsymbol{Y}}_0$ and $\hat{\boldsymbol{Y}}_1 = \hat{\boldsymbol{Y}} - \hat{\boldsymbol{Y}}_0$.

(4) Determine $ESS_{FM} = ||\boldsymbol{Y} - \hat{\boldsymbol{Y}}||^2$, $SSE_{H_0} = ||\boldsymbol{Y} - \hat{\boldsymbol{Y}}_0||^2$, $||\hat{\boldsymbol{Y}} - \hat{\boldsymbol{Y}}_0||^2$, and the F-statistic.

(5) Verify that $||\hat{\boldsymbol{Y}} - \hat{\boldsymbol{Y}}_0||^2 = \boldsymbol{Z}'[\boldsymbol{AM}^{-1}\boldsymbol{A}]^{-1}\boldsymbol{Z}$.

(6) Find $\boldsymbol{c}$ and $\boldsymbol{a}_c$ so that $||\hat{\boldsymbol{Y}} - \hat{\boldsymbol{Y}}_0||^2/S^2 = t_c^2 = (\boldsymbol{a}_c, \boldsymbol{Y})^2/[S^2||\boldsymbol{a}_c||^2]$.

(7) Let $\boldsymbol{\xi} = \boldsymbol{A\theta}$. Please express $\boldsymbol{\theta}_1 = \boldsymbol{\theta} - \boldsymbol{\theta}_0$ as a linear combination of two vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, with coefficients $(\beta_2 - \beta_3)$ and β_4.
Hint: Define $\boldsymbol{x}_5 = \boldsymbol{x}_2 + \boldsymbol{x}_3$, $\boldsymbol{x}_j^\perp = p(\boldsymbol{x}_j | V_1 = V \cap V_0^\perp)$, for $j = 2,3,4$. Show that $\boldsymbol{\theta}_1 = \beta_2\boldsymbol{x}_2^\perp + \beta_3\boldsymbol{x}_3^\perp + \beta_4\boldsymbol{x}_4^\perp$ and that $\boldsymbol{x}_2^\perp + \boldsymbol{x}_3^\perp = 0$.

(8) Express the noncentrality parameter δ as a quadratic form in $(\beta_2 - \beta_3)$ and β_4 by using the result of (7) and also by using the formula $\delta = \boldsymbol{\xi}'[\boldsymbol{AM}^{-1}\boldsymbol{A}']^{-1}\boldsymbol{\xi}/\sigma^2$.

(9) For $\boldsymbol{\beta} = (10,3,5,-2)'$, $\sigma^2 = 16$, and $\alpha = 0.05$, find $\boldsymbol{\theta}, \boldsymbol{\theta}_0, \boldsymbol{\theta}_1, \delta$, and the power of the F-test.

Note that the notations and their corresponding definitions are the same as those in your textbook (Stapleton (1995)).

Table 6 0.95 and 0.99 Quantiles of the Studentized Range Distribution for Parameters k, ν

ν		k: 2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	0.95	18.00	27.00	32.80	37.10	40.40	43.10	45.40	47.40	49.10	50.60	52.00	53.20	54.30	55.40
	0.99	90.0	135.0	164.0	186.0	202.0	216.0	227.0	237.0	246.0	253.0	260.0	266.0	272.0	277.0
2	0.95	6.09	8.30	9.80	10.90	11.70	12.40	13.00	13.50	14.00	14.40	14.70	15.10	15.40	15.70
	0.99	14.00	19.00	22.30	24.70	26.60	28.20	29.50	30.70	31.70	32.60	33.40	34.10	34.80	35.40
3	0.95	4.50	5.91	6.82	7.50	8.04	8.48	8.85	9.18	9.46	9.72	9.95	10.20	10.40	10.50
	0.99	8.26	10.60	12.20	13.30	14.20	15.00	15.60	16.20	16.70	17.10	17.50	17.90	18.20	18.50
4	0.95	3.93	5.04	5.76	6.29	6.71	7.05	7.35	7.60	7.83	8.03	8.21	8.37	8.52	8.66
	0.99	6.51	8.12	9.17	9.96	10.60	11.10	11.50	11.90	12.30	12.60	12.80	13.10	13.30	13.50
5	0.95	3.64	4.60	5.22	5.67	6.03	6.33	6.58	6.80	6.99	7.17	7.32	7.47	7.60	7.72
	0.99	5.70	6.97	7.80	8.42	8.91	9.32	9.67	9.97	10.20	10.50	10.70	10.90	11.10	11.20
6	0.95	3.46	4.34	4.90	5.31	5.63	5.89	6.12	6.32	6.49	6.65	6.79	6.92	7.03	7.14
	0.99	5.24	6.33	7.03	7.56	7.97	8.32	8.61	8.87	9.10	9.30	9.49	9.65	9.81	9.95
7	0.95	3.34	4.16	4.68	5.06	5.36	5.61	5.82	6.00	6.16	6.30	6.43	6.55	6.66	6.76
	0.99	4.95	5.92	6.54	7.01	7.37	7.68	7.94	8.17	8.37	8.55	8.71	8.86	9.00	9.12
8	0.95	3.26	4.04	4.53	4.89	5.17	5.40	5.60	5.77	5.92	6.05	6.18	6.29	6.39	6.48
	0.99	4.74	5.63	6.20	6.63	6.96	7.24	7.47	7.68	7.87	8.03	8.18	8.31	8.44	8.55
9	0.95	3.20	3.95	4.42	4.76	5.02	5.24	5.43	5.60	5.74	5.87	5.98	6.09	6.19	6.28
	0.99	4.60	5.43	5.96	6.35	6.66	6.91	7.13	7.32	7.49	7.65	7.78	7.91	8.03	8.13
10	0.95	3.15	3.88	4.33	4.65	4.91	5.12	5.30	5.46	5.60	5.72	5.83	5.93	6.03	6.11
	0.99	4.48	5.27	5.77	6.14	6.43	6.67	6.87	7.05	7.21	7.36	7.48	7.60	7.71	7.81
11	0.95	3.11	3.82	4.26	4.57	4.82	5.03	5.20	5.35	5.49	5.61	5.71	5.81	5.90	5.99
	0.99	4.39	5.14	5.62	5.97	6.25	6.48	6.67	6.84	6.99	7.13	7.25	7.36	7.46	7.56
12	0.95	3.08	3.77	4.20	4.51	4.75	4.95	5.12	5.27	5.40	5.51	5.62	5.71	5.80	5.88
	0.99	4.32	5.04	5.50	5.84	6.10	6.32	6.51	6.67	6.81	6.94	7.06	7.17	7.26	7.36
13	0.95	3.0-6	3.73	4.15	4.45	4.69	4.88	5.05	5.19	5.32	5.43	5.53	5.63	5.71	5.79
	0.99	4.26	4.96	5.40	5.73	5.98	6.19	6.37	6.53	6.67	6.79	6.90	7.01	7.10	7.19
14	0.95	3.03	3.70	4.11	4.41	4.64	4.83	4.99	5.13	5.25	5.36	5.46	5.55	5.64	5.72
	0.99	4.21	4.89	5.32	5.63	5.88	6.08	6.26	6.41	6.54	6.66	6.77	6.87	6.96	7.05

Table 6 *(contd)* 0.95 and 0.99 Quantiles of the Studentized Range Distribution for Parameters k, ν

ν		k: 2	3	4	5	6	7	8	9	10	11	12	13	14	15
15	0.95	3.01	3.67	4.08	4.37	4.60	4.78	4.94	5.08	5.20	5.31	5.40	5.49	5.58	5.65
	0.99	4.17	4.83	5.25	5.56	5.80	5.99	6.16	6.31	6.44	6.55	6.66	6.76	6.84	6.93
16	0.95	3.00	3.65	4.05	4.33	4.56	4.74	4.90	5.03	5.15	5.26	5.35	5.44	5.52	5.59
	0.99	4.13	4.78	5.19	5.49	5.72	5.92	6.08	6.22	6.35	6.46	6.56	6.66	6.74	6.82
17	0.95	2.98	3.63	4.02	4.30	4.52	4.71	4.86	4.99	5.11	5.21	5.31	5.39	5.47	5.55
	0.99	4.10	4.74	5.14	5.43	5.66	5.85	6.01	6.15	6.27	6.38	6.48	6.57	6.66	6.73
18	0.95	2.97	3.61	4.00	4.28	4.49	4.67	4.82	4.96	5.07	5.17	5.27	5.35	5.43	5.50
	0.99	4.07	4.70	5.09	5.38	5.60	5.79	5.94	6.08	6.20	6.31	6.41	6.50	6.58	6.65
19	0.95	2.96	3.59	3.98	4.25	4.47	4.65	4.79	4.92	5.04	5.14	5.23	5.32	5.39	5.46
	0.99	4.05	4.67	5.05	5.33	5.55	5.73	5.89	6.02	6.14	6.25	6.34	6.43	6.51	6.58
20	0.95	2.95	3.58	3.96	4.23	4.45	4.62	4.77	4.90	5.01	5.11	5.20	5.28	5.36	5.43
	0.99	4.02	4.64	5.02	5.29	5.51	5.69	5.84	5.97	6.09	6.19	6.29	6.37	6.45	6.52
24	0.95	2.92	3.53	3.90	4.17	4.37	4.54	4.68	4.81	4.92	5.01	5.10	5.18	5.25	5.32
	0.99	3.96	4.54	4.91	5.17	5.37	5.54	5.69	5.81	5.92	6.02	6.11	6.19	6.26	6.33
30	0.95	2.89	3.49	3.84	4.10	4.30	4.46	4.60	4.72	4.83	4.92	5.00	5.08	5.15	5.21
	0.99	3.89	4.45	4.80	5.05	5.24	5.40	5.54	5.65	5.76	5.85	5.93	6.01	6.08	6.14
40	0.95	2.86	3.44	3.79	4.04	4.23	4.39	4.52	4.63	4.74	4.82	4.91	4.98	5.05	5.11
	0.99	3.82	4.37	4.70	4.93	5.11	5.27	5.39	5.50	5.60	5.69	5.77	5.84	5.90	5.96
60	0.95	2.83	3.40	3.74	3.98	4.16	4.31	4.44	4.55	4.65	4.73	4.81	4.88	4.94	5.00
	0.99	3.76	4.28	4.60	4.82	4.99	5.13	5.25	5.36	5.45	5.53	5.60	5.67	5.73	5.79
120	0.95	2.80	3.36	3.69	3.92	4.10	4.24	4.36	4.48	4.56	4.64	4.72	4.78	4.84	4.90
	0.99	3.70	4.20	4.50	4.71	4.87	5.01	5.12	5.21	5.30	5.38	5.44	5.51	5.56	5.61
Infinity	0.95	2.77	3.31	3.63	3.86	4.03	4.17	4.29	4.39	4.47	4.55	4.62	4.68	4.74	4.80
	0.99	3.64	4.12	4.40	4.60	4.76	4.88	4.99	5.08	5.16	5.23	5.29	5.35	5.40	5.45