第一卷

Mathematical Statistics: Ph.D. Qualifier Exam September 12, 2005

(100 minutes)

1. (22pts). Suppose that $F(t) = 1$ and $F(t - \epsilon) < 1$ for some t and for all $\epsilon > 0$ and that F is differentiable at all values less than t with derivative f such that $\lim_{x \uparrow t} f(x) = c$ with $0 < c < \infty$, and F is continuous at t. Let $X_1, \ldots, X_n$ be IID with CDF F and let $X_{(n)} = \max(X_1, \ldots, X_n)$. Please find a_n and b_n such that $a_n(X_{(n)} - b_n)$ converges in distribution to a nondegenerate distribution as $n \to \infty$. Please also give the nondegenerate distribution.

2. Suppose that $X \sim \mathcal{P}oi(\theta)$ and that we are trying to estimate $e^{r\theta}$, where r is a fixed nonzero real number.
   (a) (10pts). Please find the Cramer-Rao lower bound for unbiased estimators.
   (b) (12pts). Please find the UMVUE.
   (c) (4pts). Is there any unbiased estimator achiving the Cramer-Rao lower bound? Prove or disprove your answer.

3. Suppose that $X \sim \mathcal{N}(\theta, 1)$ given θ, and let θ have an $\mathcal{N}(0, 1)$ prior. Suppose that the parameter space and the action space are both $(-\infty, \infty)$. Let loss $L(\theta, a) = \begin{cases} 0, & \text{if } a \geq \theta \\ 1, & \text{if } a < \theta \end{cases}$.
   (a) (12pts). Please comput the Bayes risk for a decision rule $\delta(X)$.
   (b) (6pts). Please show that there is no Bayes rule.
   (c) (6pts). Please show that every decision rule is inadmissible.

4. Let $\mathcal{P} = \{P_\theta : \theta \in \Theta\}$ be a parameteric family. For each $0 \leq \alpha \leq 1$, let $\phi_\alpha(x) = 1_{S_\alpha}(x)$ be a size α test of H$_0$: $\theta = \theta_0$ verse H$_1$: $\theta \in \Theta / \{\theta_0\}$ such that, for $0 < \alpha < 1$, $S_\alpha = \cap_{\gamma > \alpha} S_\gamma$. Suppose that $S_0 = \emptyset$ and $S_1 = \mathcal{X}$. Define the P-value of the observed data x by $p_{H_0}(x) = \inf\{\varphi(\gamma) : \phi_\gamma(x) = 1\}$, where $\varphi(\gamma)$ is the size of the test ϕ_γ.
   (a) (5pts). Prove that if $0 \leq \alpha < \beta \leq 1$, then $S_\alpha \subseteq S_\beta$.
   (b) (15pts). Please show that, given $X \sim P_{\theta_0}$, $p_{H_0}(X) \sim$Unif$(0, 1)$.
   (c) (8pts). Suppose that ϕ_α is unbiased for each α. Prove that, for all $\theta \neq \theta_0$, $P_\theta(p_{H_0}(X) \leq \alpha) \geq P_{\theta_0}(p_{H_0}(X)) \leq \alpha)$. In words, P-value is likely smaller under the alternative than under the null.

第二卷

P1.

Ph.D. Qualifying Exam

1. [10 %] Let $X_1, X_2, \ldots, X_n$ be i.i.d. from $N(\theta, 1)$, $-\infty < \theta < \infty$. Define
$$\hat{\theta}_n = \begin{cases} \bar{X} & \text{if } |\bar{X}| \geq n^{-1/4} \\ t\bar{X} & \text{if } |\bar{X}| < n^{-1/4}, \end{cases}$$
where t is a fixed constant. Show that $\hat{\theta}$ possesses the asymptotic normality.

2. [10 %] Let X be a single observation having p.d.f. $\frac{1}{2}(1-\theta^2)e^{\theta x - |x|}$, $|\theta| < 1$. Show that if $0 \leq \alpha \leq 1/2$, then $\alpha X + \beta$ is admissible for estimating $E(X)$ under the squared error loss.

3. [10 %] Let $(Y_1, Z_1), \ldots, (Y_n, Z_n)$ be i.i.d. with p.d.f.
$$f(x, y|\lambda, \mu) = \lambda^{-1}\mu^{-1}e^{-y/\lambda}e^{-z/\mu}I_{(0,\infty)}(y)I(0,\infty)(z),$$
where $\lambda, \mu > 0$. Suppose that we only observe $X_i = \min(Y_i, Z_i)$ and $\delta_i = 1$ if $X_i = Y_i$ and $\delta_i = 0$ if $X_i = Z_i$. Find the MLE (maximum likelihood estimator) of (λ, μ).

4. [10 %] Let $X_1, X_2, \ldots, X_n$ be i.i.d. from the uniform distribution $U(0, \theta)$, where $\theta > 0$ is unknown. Construct a confidence band for the c.d.f. of X_1 with confidence coefficient $1 - \alpha$.

5. [10 %] Let $X_1, X_2, \ldots, X_n$ be i.i.d. from the uniform distribution on $(\mu - 1/2, \mu + 1/2)$, where $-\infty < \mu < \infty$ is unknown. Show that $(X_{(1)} + X_{(n)})/2$ is an MRIE (minimum risk invariant estimator) of μ if the loss function is convex and even, where $X_{(1)}$ and $X_{(n)}$ represent the smallest and the largest observation in the sample, respectively.

6. [10 %] Let $X_{i1}, X_{i2}, \ldots, X_{in_i}, i = 1, 2, n_i \geq 2$ be two independent samples i.i.d. from $N(\mu_i, \sigma_i^2)$, $i = 1, 2$, respectively, where μ_i and $\sigma_i^2 > 0$ are unknown, but $\sigma_2^2/\sigma_1^2 = \lambda$. Find a UMPU (uniformly most powerful unbiased) size α test for testing $H_0 : \lambda = \lambda_0$ versus $\lambda \neq \lambda_0$.

7. [10 %] Suppose that X has a $N(\theta, 1)$ distribution. Find the generalized Bayes rule with respect to the prior $\pi(\theta) = 1$, $-\infty < \theta < \infty$ under the loss function $L(\theta, a) = (\theta a - 1)^2$ and show that it is admissible.

8. [10 %] Let $X_1, X_2, \ldots, X_n$ be i.i.d. with common continuous distribution function F and p.d.f. f. We want to test $H_0 : F(x) = G(x)$ against $H_1 : F(x) = G^{1+\delta}(x)$, where G is a specified continuous distribution function with p.d.f. g. Derive a UMP size α test for testing $H_0 : \delta \leq 0$ against $H_1 : \delta > 0$.

9. [10 %] Let $X_1, X_2, \ldots, X_n$ be i.i.d. from $p \in \mathcal{P}$ containing all symmetric c.d.f. with finite means and with Lebesgue p.d.f.'s on $\mathcal{R}$. Prove that there is no UMVUE of $\mu = E(X_1)$.

10. [10 %] Let $X_1, X_2, \ldots, X_n$ be i.i.d. with mean μ and variance $\sigma^2 > 0$. Consider the test, in testing $H_0 : \mu \leq \mu_0$ versus $H_1 : \mu > \mu_0$,

$$T_{c_\alpha}(X_1, \ldots, X_n) = 1 \text{ if and only if } \sqrt{n}(\bar{X} - \mu_0)/S > c_\alpha = t_{n-1;1-\alpha},$$

where $\bar{X}$ and S^2 are the usual sample mean and sample variance, respectively, and $t_{n-1;1-\alpha}$ is the $(1-\alpha)100^{th}$ percentile of the t distribution of $n-1$ degrees of freedom. Show that T_{c_α} has asymptotic significance level α and is consistent.

第三卷

PhD QR exam 94 MathStat

1. (25 points) Let $Z \sim N(0,1)$ and g be a real-valued function.
(a) (15 points) Show that $E[Zg(Z)] = E[g'(Z)]$ under some regularity conditions.
(b) (10 points) What are these regularity conditions?

2. (25 points) Consider a stationary autoregressive process of order 1,

$$Y_t = \theta Y_{t-1} + \sigma e_t, \quad t = 0, \pm 1, ...,$$

where e_t are independent standard normal random variables and $|\theta| < 1$ and $\sigma^2 > 0$ are unknown parameters. Suppose that $Y_1, ..., Y_{1+n}$ are observed.
(a) (5 points) Derive the likelihood function based on these observations.
(b) (10 points) Let $\hat{\theta}_n$ and $\hat{\sigma}_n$ denote the maximum likelihood estimators of θ and σ. Find $\hat{\theta}_n$ and $\hat{\sigma}_n$.
(c) (10 points) Show that $\hat{\theta}_n$ is invariant and $\hat{\sigma}_n$ is equivariant under scale transformations.

3. (30 points) Suppose that Y_n follows $N(0,1)$ with probability π_n and $N(0, \tau_n^2)$ with probability $1 - \pi_n$, where $\pi_n \to \pi$ and $\tau_n \to \infty$. Let F_n denote the cdf of Y_n and define $G(y) = \lim_{n\to\infty} F_n(y)$.
(a) (15 points) Find $G(y)$ and show that G is a distribution function iff $\pi = 1$.
(b) (10 points) Find $\text{Var}(Y_n)$ and $\lim_{n\to\infty} \text{Var}(Y_n)$.
(c) (5 points) Suppose that $\pi = 1$. Does the limit of variance agree with the variance of the limit distribution?

4. (20 points) Let f_0 and f_1 be two pdf's w.r.t. the Lebesgue measure on $\Re$; define

$$f_{i,\mu,\sigma}(x) = \frac{1}{\sigma} f_i\left(\frac{x-\mu}{\sigma}\right), \quad x \in \Re$$

for $\theta = (i, \mu, \sigma) \in \Omega \equiv \{0,1\} \times \Re \times \Re^+$. Given an *i.i.d.* sample $X_1, ..., X_n$ from $f_{i,\mu,\sigma}$ for some unknown parameter $\theta = (i, \mu, \sigma) \in \Omega$, consider the hypotheses:

$$H_0 : i = 0, 0 < \sigma < \infty, -\infty < \mu < \infty \text{ vs } H_1 : i = 1, 0 < \sigma < \infty, -\infty < \mu < \infty.$$

(a) (10 points) Show that the statistical model and the hypotheses are invariant under the affine linear group

$$G = \{g : gx = (ax_1 + b, ..., ax_n + b), (b, a) \in \Re \times \Re^+\}.$$

(b) (10 points) Give a level α UMPI test.