

考試科目 Course	統計系, 碩士班 資格考 (數統)	開課系級 Dept. & Class	日期 Date, Period	2 月 18 日 第 節	試題編號 Course No.	2
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1. Let X, Y be random variables. Consider the following 3 statements

(a) X, Y are independent

(b) $E(XY) = E(X)E(Y)$

(c) $E(Y|X) = E(Y)$

Determine the correct orderings for these statements (such as (a) \Rightarrow (b) \Rightarrow (c)

but (b) $\not\Rightarrow$ (a), (c) $\not\Rightarrow$ (b) — This is not necessarily the correct answer.)

Prove the orderings and give counter examples to show the reverse implications do not hold.

2. Let $(X_1, Y_1), \dots, (X_n, Y_n)$ be i.i.d. bivariate normal $N(0, \Sigma)$,

where $\Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$

(a) Consider the r.v. $Z_n = \frac{1}{n} \sum_{i=1}^n X_i Y_i$, and determine mean and variance of Z_n .

(Note: You might want to use the fact that for r.v.'s U and V , $\text{Var}(U) = E[\text{Var}(U|V)] + \text{Var}[E(U|V)]$.)

(b) Using an appropriate conditioning argument, or otherwise, determine that the distribution of the r.v. $(W_n - \rho)$, where

$$W_n = \frac{\sum_{i=1}^n X_i Y_i}{\sum_{i=1}^n X_i^2} \text{ is } ((1-\rho^2)/n)^{1/2} T_n,$$

where T_n denotes a T -distribution with n degrees of freedom.

(c) Determine the mean and variance of W_n in (b), and determine the relative efficiency of W_n to Z_n as an estimator of ρ .

(Note: T_n possesses variance $\frac{n}{n-2}$.)

(d) Are either of the estimators W_n or Z_n asymptotically efficient for ρ as $n \rightarrow \infty$? Explain!