95.02.17

第一卷

數理統計95年

1. (20 pts) Let X be a poisson random variable with parameters λ and cdf $F(x)$.
   Let Z be a standard normal random variable with cdf $\Phi(z)$.
   Find the distributions of the random variables $F(X)$ and $\Phi(Z)$.

2. (20 pts) Let $F(x)$ be a cdf of a nonnegative random variable X, i.e. $F(x) = 0$ if $x < 0$. Assume that F(x) has only countable discontinuous point and may not be differentiable everywhere. Find EX and Var X in terms of $F(x)$ only.

3. (20pts) Let X_i, $i = 1, \ldots, n$ be a random sample from a positive value population with mean μ variance $\sigma^2 < \infty$ and $E(X_i - \mu)^4 < \infty$.
   Let $S_n^2 = \frac{1}{n(n-1)} \sum_{i<j} (X_i - X_j)^2$. Show that S_n converges in probability to σ.
   $S_n \xrightarrow{P} \sigma$.

4. (20%) Let X_i, $i = 1, \ldots,$ n be a random sample from a exponential with mean λ.
   Let $T_n = T_n(X_1, \ldots, X_n)$ be a mle of λ^2. Define $T_n^{(i)} = T_{n-1}(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n)$.
   Let $JK(T_n) = nT_n - \frac{n-1}{n} \sum_{i=1}^{n} T_n^{(i)}$. Is $JK(T_n)$ a better estimator of λ^2? Justify your answer. Is there a best estimator of λ^2? If yes, find it. If not, give you reason.

5. (20%) Let (X_i, Y_i) $i = 1, 2, \ldots, n$ be a random sample from a bivariate normal with unknown parameters μ_X, μ_Y, σ_X^2, σ_Y^2 and ρ. Is there an UMP unbiased level α test for the testing H_0: $\mu_X = \mu_Y$ vs. H_1: $\mu_X \neq \mu_Y$? Show your answer.

4. Suppose that $X_1, X_2, \ldots, X_n$ are iid with a $beta(\mu,1)$ pdf and $Y_1, Y_2, \ldots, Y_m$ are iid with a $beta(\theta,1)$ pdf. Also assume that X s are independent of the Y s. (20%)

(a) Find an LRT of $H_0 : \theta = \mu$ versus $H_1 : \theta \neq \mu$.

(b) Show that the test in part (a) can be based on the statistic

$$T = \frac{\sum_{i=1}^{n} \log X_i}{\sum_{i=1}^{n} \log X_i + \sum_{i=1}^{m} \log Y_i}.$$

(c) Find the distribution of T when H_0 is true, and then show how to get a test of size $\alpha = 0.1$.

第二卷

PhD QR exam 95.02.17 MathStat

1. (30 points) (a) (10 points) Let $X_1, ..., X_n$ be i.i.d. Unif$(0, \theta)$ random variables. Show that $T = X_{(n)}$, the largest order statistic, is complete and sufficient for θ.
(b) (10 points) Let $X_1, ..., X_n$ be i.i.d. Unif$(\theta - \frac{1}{2}, \theta + \frac{1}{2})$ random variables. Show that $T = (X_{(1)}, X_{(n)})$ is minimal sufficient, but is not complete. Here $X_{(1)}$ is the smallest order statistic.
(c) (10 points) Show that if a minimal sufficient statistic exists, a necessary condition for a sufficient statistic to be complete is for it to be minimal.

2. (25 points) Let $X_i, i = 1, ..., r, (r > 2)$ be independent, with distributions $N(\theta_i, 1)$ and let the estimator $\delta_c = (\delta_{1c}, ..., \delta_{rc})'$ of $\theta = (\theta_1, ..., \theta_r)'$ be given by

$$\delta_{ic} = (1 - c\frac{r-2}{S^2})X_i, \; S^2 = \sum X_j^2.$$

(a) (15 points) Show that the risk function of δ_c, with loss function $L(\theta, d) = \frac{1}{r}\sum(d_i - \theta_i)^2$, is

$$R(\theta, \delta_c) = 1 - \frac{(r-2)^2}{r} E_\theta(\frac{2c - c^2}{S^2}).$$

(b) (5 points) Show that the estimator δ_c dominates X provided $0 < c < 2$ and $r \geq 3$. (Note that X equals δ_c with $c = 0$.)
(c) (5 points) Show that the James-Stein estimator δ, which equals δ_c with $c = 1$, dominates all estimators δ_c with $c \neq 1$.

3. (30 points) Let $(X_i, Y_i), i = 1, ..., n$ be iid according to the bivariate normal distribution with parameter $(\mu_x, \mu_y, \sigma_x, \sigma_y, \rho)$. Let R denote the sample correlation coefficient, $\frac{\sum(X_i - \bar{X})(Y_i - \bar{Y})}{\sqrt{\sum(X_i - \bar{X})^2 \sum(Y_i - \bar{Y})^2}}$, and consider the hypothesis testing problem $H_0 : \rho = 0$ v.s. $H_1 : \rho \neq 0$.
(a) (5 points) Show that the distribution of R does not depend on $\mu_x, \mu_y, \sigma_x, \sigma_y$, but only on ρ.
(b) (15 points) Suppose that $\rho = 0$. First show that the conditional distribution of $T = \sqrt{n-2}R/\sqrt{1-R^2}$ given $x_1, ..., x_n$ has the t-distribution with $n-2$ degrees of freedom provided $\sum(x_i - \bar{x})^2 > 0$. Then argue that this is therefore also the unconditional distribution of T. [Hint: Let $v_i = (x_i - \bar{x})/\sqrt{\sum(x_j - \bar{x})^2}$ so that $\sum v_i = 0, \sum v_i^2 = 1$. Then, rewrite T in terms of v and Y and make use of an orthogonal transformation from $(Y_1, ..., Y_n)$ to $(Z_1, ..., Z_n)$ such that $Z_1 = \sqrt{n}\bar{Y}, Z_2 = \sum v_i Y_i$.]
(c) (10 points) Show that the UMP unbiased test rejects H_0 when

$$|R|/\sqrt{(1-R^2)/(n-2)} > K_0,$$

and determine K_0 if the significance level is α.

4. (15 points) Let $X_1, ..., X_m$ and $Y_1, ..., Y_n$ be independent samples from $N(\xi, 1)$ and $N(\eta, 1)$, and consider the hypothesis $H_0 : \eta \leq \xi$ against $H_1 : \eta > \xi$. Find a UMP test.

第三卷

Mathematical Statistics: Ph.D. Qualifier Exam February 17, 2006 (100 minutes)

1. Let the distribution of $\{X_i\}_{i=1}^{\infty}$ be that of i.i.d. $\mathcal{N}(\mu,\sigma^2)$. Define $A_n = \frac{1}{n}\Sigma_{i=1}^{n} X_i$ and $B_n = \frac{1}{n}\Sigma_{i=1}^{n} X_i^2$.

   (a) (15pts). Please find the asymptotic joint distribution of A_n and B_n. That is , find a_n, b_n, and c_n so that $c_n \begin{pmatrix} A_n - a_n \\ B_n - b_n \end{pmatrix}$ converges in distribution to a nondegenerate distribution.

   (b) (15pts). Let $\bar{X}_n = \frac{1}{n}\Sigma_{i=1}^{n} X_i$ and $S_n = \sqrt{\frac{1}{n-1}\Sigma_{i=1}^{n}(X_i - \bar{X}_n)^2}$. Please find the asymptotic distribution of $\bar{X}_n + 1.96 S_n$.

2. (20pts). Let $X_1, \ldots, X_n$ be i.i.d. Ber(θ). Suppose that we have a loss function $L(\theta, a) = (\theta - a)^2/(\theta^2(1-\theta)^5)$, where the action space is $\mathcal{A} = [0, 1]$. The prior distribution of θ is Beta(α,β). Please find conditions on α and β such that the Bayes rule exists and has finite risk. Please also give the Bayes rule.

3. (20pts). Suppose that X and Y are independently distributed with $\mathcal{N}(\theta$,1) distribution. If $X \leq 3$, we set $Z = X$. If $X > 3$, we compute $Z = (X + Y)/2$. Is Z an unbiased estimator of θ? Please prove or disprove your answer. (Hint: You can start with E($Z|X$).)

4. Let $X_1, \ldots, X_n$ be i.i.d. Unif($\theta-1/2$,$\theta+1/2$) whose pdf is denoted by $f_\theta(x) = \mathbf{1}_{\theta-1/2 \leq x \leq \theta+1/2}$. Let H$_0$: $\theta = 0$ and H$_1$: $\theta \neq 0$. Assume $0 < \alpha < 1$ for the following questions.

   (a) (5pts). Suppose ϕ is a test with size α. Please prove that ϕ is unbiased level α if $\phi(x) = 1$ for all x which satisfy $f_0(x) = 0$.

   (b) (10pts). Consider the test $\phi_1(x) = \mathbf{1}_{x<-1/2} + \mathbf{1}_{1/2-\alpha<x}$. Please show that ϕ_1 is unbiased level α. Also please plot the power function.

   (c) (15pts). Prove that there does not exist a UMPU (uniformly most power unbiased) level α test. (Hint: You can consider ϕ_1 and another similar test. Then prove it is impossible for a single test to achive both maxima.)