2003/09/30
數理統計

Mathematical Statistics: Ph.D. Qualifier September 30, 2003

1. Suppose taht $\{X_n\}_{n=1}^{\infty}$ are conditionally IID $U(0,\theta)$ random variables given θ. The parameter space is the interval $[1,2]$. Suppose that we only get to observe $Y_i = I_{[0,1]}(X_i)$ for each i. That is, we only see whether or not each observation is between 0 and 1.

   (a) (8pts). find the MLE of θ based on $Y_1,\ldots,Y_n$.

   (b) (8pts). Give the asymptotic (as $n \to \infty$) distribution of the MLE found in (a).

   (c) (8pts). Find the MLE of θ based on $X_1,\ldots,X_n$.

   (d) (8pts). Give the asymptotic (as $n \to \infty$) distribution of the MLE found in (c).

   (e) (4pts). In terms of asymptotic efficiency, how does the MLE found in (a) compare to the MLE based on the actual X_i values?

2. Suppose that $X \sim N(\theta,1)$ given θ.

   (a) (8pts). Find the UMVUE of θ^2 based on X.

   (b) (4pts). What is wrong with the estimator in (a)?

   (c) (10pts). Suppose that we have a decision problem with loss function $L(\theta,a) = (\theta^2 - a)^2$. Find the generalized Bayes rule with respect to a Lebesgue measure $(\pi(\theta) = 1)$.

   (d) (8pts). Show that the bayes estimator in (c) is inadmissible.

3. Let $f_\theta(x) = \begin{cases} \frac{1}{2\theta} & \text{if } 0 < x \le \theta \\ \frac{\theta}{2x^2} & \text{if } \theta < x \end{cases}$ and $X \sim f_\theta(x)$.

   (a) (10pts). Show that this family of distributions has MLR.

   (b) (12pts). Let $0 < \alpha < 0.5$. Find an UMP level α test of $H_0 : \theta = \theta_0$ versus $H_1 : \theta > \theta_0$ and show that it is unique. (Hint: the proof is the same as that of Neyman-Pearson Lemma).

   (c) (12pts). If $0.5 < \alpha < 1$, find two UMP level α test of $H_0 : \theta = \theta_0$ versus $H_1 : \theta > \theta_0$ that are not almost surely equal.

A

2003 年 9 月博士班資格考試（數理統計）

1. When a Poisson process with rate λ is observed for a time interval of length τ, the number X of events occurring has the Poisson distribution $P(\lambda\tau)$. Under an alternative scheme, the process is observed until r events have occurred, and the time T of observation is then a random variable such that $2\lambda T$ has a χ^2-distribution with $2r$ degrees of freedom. For testing $H:\lambda \le \lambda_0$ at level α one can, under either design, obtain a specified power β against an alternative λ_1 by choosing τ and r sufficiently large.
   (i) The ratio of the time of observation required for this purpose under the first design to the expected time required under the second is $\lambda\tau / r$.
   (ii) Determine for which values of λ each of the two designs is preferable when $\lambda_0 = 1, \lambda_1 = 2, \alpha = 0.05, \beta = 0.9$.
   (25%)

2. Let $X_{1,\dots,}X_m$ be a random sample from a N(μ,σ^2) population. Let $Y_{1,\dots,}Y_n$ be a random sample from a N (θ,τ^2) population. Suppose the two samples are independent. All four parameters are unknown. Consider testing $H:\mu \le 0$ or $\theta \le 0$ versus $K:\mu > 0$ and $\theta > 0$. Consider only tests based on the sufficient statistics $\overline{X}, S^2, \overline{Y}$ and T^2, the sample means and variances. Consider the group of transformations with elements defined by

   $g_{c,d}(\bar{x}, s^2, \bar{y}, t^2) = (c\bar{x}, c^2 s^2, d\bar{y}, d^2 t^2)$, where c and d are any positive constants.

   (You do not need to show that these transformations form a group.)

   (i) Show that this hypothesis testing problem is invariant under this group.
   (ii) Find a maximal invariant fro this group.
   (iii) For a fixed α, find one invariant, nonrandomized level-α test for this problem.
   (30%)

3. Let $X_{1,\dots,}X_m$ be i.i.d. random variables having the beta distribution $B(\beta,\beta)$ with an unknown $\beta > 0$. Find a minimal sufficient statistic for β.
(20%)

B

ESTIMATION AND TESTING HYPOTHESES

Ph. D. Examination-September 30, 2003

1. Let $X_1, X_2, \cdots, X_n$ be independent normal with common mean μ and common variance σ^2. Determine the maximum likelihood estimates of μ and σ in each of the following cases.

$$(a) \quad -\infty < \mu < \infty, \quad 0 < \sigma < \infty$$

$$(b) \quad a_1 < \mu < a_2, \quad 0 < \sigma < \infty,$$

where $-\infty < a_1 < a_2 < \infty$.

2. Let $X_1, X_2, \cdots, X_n$ be independent normal with common mean μ and common variance 1, and let

$$\gamma(\mu) = \text{Prob } (X_j < 0) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{-\mu} e^{-x^2/2} dx.$$

(a) Show that there exists an unbiased estimator of $\gamma(\mu)$.

(b) Determine an unbiased estimate of $\gamma(\mu)$ which is a function of the sample mean X only.

(c) Does the estimate in (b) have minimum variance?

3. Let $X_1, X_2, \cdots, X_n$ be independent with common density function

$$g(x) = \begin{cases} e^{\theta - x} & \text{if } x > \theta \\ 0 & \text{if } x \leq \theta. \end{cases}$$

Determine the following tests of level α for testing the hypothesis that $\theta = 0$, and obtain their power functions.

(a) a uniformly most powerful test against $0 < \theta < \infty$,

(b) a uniformly most powerful test against $-\infty < \theta < 0$.