第一卷

Qualifying Exam (Linear Models)

10/8/2002

1. Consider $\mathbf{y}=\mathbf{X}\boldsymbol{\beta}+\boldsymbol{\varepsilon}$, with $\boldsymbol{\varepsilon}\sim N_n(\mathbf{0},\sigma^2\mathbf{V})$ and $\mathbf{V}=\mathbf{I}_n+\mathbf{J}_n$, where $\mathbf{J}_n$ is an $n\times n$ matrix of ones. Let $\hat{\boldsymbol{\beta}}=(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$, and $\tilde{\boldsymbol{\beta}}=(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{V}^{-1}\mathbf{y}$.

   (a) What is the distribution of $Q(\hat{\boldsymbol{\beta}})=(\mathbf{y}-\mathbf{X}\hat{\boldsymbol{\beta}})'(\mathbf{y}-\mathbf{X}\hat{\boldsymbol{\beta}})$?

   (b) Are $\tilde{\boldsymbol{\beta}}$ and $Q(\hat{\boldsymbol{\beta}})$ independent?

2. Consider the model $\mathbf{y}=\mathbf{X}\boldsymbol{\beta}+\boldsymbol{\varepsilon}=[\mathbf{X}_1\,|\,\mathbf{X}_2]\begin{pmatrix}\boldsymbol{\beta}_1\\ \boldsymbol{\beta}_2\end{pmatrix}+\boldsymbol{\varepsilon}$, with $\boldsymbol{\varepsilon}\sim N_n(\mathbf{0},\sigma^2\mathbf{I}_n)$, where $\mathbf{X}_i$ is an $n\times p_i$ matrix of full column rank, $i=1,2$. Let $\breve{\boldsymbol{\beta}}_1=(\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1'\mathbf{y}$ and $\hat{\boldsymbol{\beta}}=\begin{pmatrix}\hat{\boldsymbol{\beta}}_1\\ \hat{\boldsymbol{\beta}}_2\end{pmatrix}=(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$.

   (a) Find $\operatorname{cov}(\breve{\boldsymbol{\beta}}_1)$ and $\operatorname{cov}(\hat{\boldsymbol{\beta}}_1)$.

   (b) Show that $\operatorname{var}(\ell'\breve{\boldsymbol{\beta}}_1)\le\operatorname{var}(\ell'\hat{\boldsymbol{\beta}}_1)$.

第二卷

1. Consider the one-way model: $y_{ij} = \eta_i + \varepsilon_{ij}$, $j = 1, \ldots, n_i$, $i = 1, \ldots, I$.

Error terms independently follow normal distribution with mean 0 and $\mathrm{Var}(\varepsilon_{ij}) = \sigma^2 \lambda_i$, where λ_i 's are given.

(1) Show that the best linear unbiased estimator (BLUE) of $\eta_1, \ldots, \eta_I$ are the same as if $\lambda_i = 1, \forall i$. (15 pts)

(2) Show that the conventional estimator of σ^2 becomes

$$\frac{\sum\sum (y_{ij} - \bar{y}_{i.})^2 / \lambda_i}{n - I}, \text{ where } n = \sum n_i. \quad \text{(15 pts)}$$

(3) If we thoughtlessly use $\dfrac{\sum\sum (y_{ij} - \bar{y}_{i.})^2}{n - I}$ to estimate σ^2, what is the expectation of this estimator? Is it unbiased? (15 pts)

2. For the two-way model without replicates,
$y_{ij} = \mu + \tau_i + \beta_j + \varepsilon_{ij}$, $i = 1, \ldots, I$, $j = 1, \ldots, J$,
where $\sum \tau_i = 0$, $\sum \beta_j = 0$, and $\varepsilon_{ij} \sim N(0, \sigma^2)$ independently.

Find test statistics and their distributions for the following null hypotheses:

(1) $\tau_1 = \ldots = \tau_I = 0$. (30 pts)

(2) $\tau_1 = \beta_1$. (25 pts)

第 三 卷

1. Consider the data of percent scores on homework (x_1), midterm (x_2) and the final (Y) from a statistics class. Suppose the following multiplicative model

$$Y = \gamma_0 + \gamma_3(x_1 - \gamma_1)(x_2 - \gamma_2) + \varepsilon$$

is considered to be more adequate than the usual additive model

$$Y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \varepsilon$$

10% (a) Describe in words and/or pictures, the difference between the additive model and the multiplicative model. In particular, how does $E(Y)$ change in each case as a function of x_1 for two different values of x_2?

10% (b) Rewrite the multiplicative model into a linear model (re-parameterization) and describe how to test whether the multiplicative model significantly improves the fit of the additive model.

2. Consider the following general linear model

$$Y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \varepsilon$$

10% (a) Find a reduced model for testing the following hypothesis

$$H: \begin{pmatrix} 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

15% (b) Find an F-statistic (a general expression) for testing the above hypothesis.

3. Consider two parameters θ_1 and θ_2 and assume that their least squares estimators $\hat{\theta}_1$ and $\hat{\theta}_2$ are independent normal with mean θ_1 and θ_2 and variance $\sigma_1^2 = 4$ and $\sigma_2^2 = 9$ respectively.

10% (a) Find a test statistic for testing the hypothesis

$$H: \theta_1 = \theta_{10}, \theta_2 = \theta_{20}$$

where θ_{10} and θ_{20} are known constants.

10% (b) Find a set of Bonferroni confidence intervals for θ_1 and θ_2 such that the overall confidence level is $1-\alpha$.

4. Consider the one-way analysis of covariance model with different slpoes

$$Y_{ij} = \theta_i + \beta_i x_{ij} + \varepsilon_{ij} \quad i = 1,\ldots,k, j = 1,\ldots,n$$

where $\varepsilon_{11},\ldots,\varepsilon_{kn}$ are iid normal with mean 0 and unknown variance σ^2

15% (a) Find the least squares estimators for the parameters $\theta_i, \beta_i, \sigma^2$

5% (b) Are the least squares estimators of $\beta_1,\ldots,\beta_k$ independent? Why ?

15% (c) Find a set of simultaneous confidence intervals for all contrasts of the slopes

$$\sum_{i=1}^{k} c_i \beta_i$$

with $\sum_{i=1}^{k} c_i = 0$ such that the overall confidence level is $1-\alpha$.