Mathematical Statistics: Ph.D. Qualifier April 17, 2002

1. Let $\Omega = (0, \infty) \times (0, \infty)$. Suppose that X_1, X_2, and X_3 are i.i.d. Unif(α, β), where $(\alpha, \beta) \in \Omega$ and $\alpha < \beta$. The mean parameter $\theta = (\alpha + \beta)/2$ is of interest. Suppose we have a loss function $L(\theta, a) = (\theta - a)^2$. Let $\delta_0(X) = \bar{X} = \frac{X_1+X_2+X_3}{3}$.

   (a) Find a two-dimensional sufficient statistic, T.

   (b) Use the Rao-Blackwell Theorem to find a rule $\delta_1(X)$ whose risk function is at least as good as that of $\delta_0(X)$. (Hint: $E(\bar{X}|X_{(1)}, X_{(3)}) = \frac{1}{3}E(X_{(1)} + X_{(2)} + X_{(3)}|X_{(1)}, X_{(3)})$ and $X_{(2)}|(X_{(1)}, X_{(3)}) \sim \mathrm{Unif}(X_{(1)}, X_{(3)}))$

   (c) Find the risk functions $R(\theta, \delta_0)$ and $R(\theta, \delta_1)$ and show that $R(\theta, \delta_1) < R(\theta, \delta_0)$. (Hint: $E(X_1) = \frac{1}{2}(\alpha+\beta), Var(X_1) = \frac{1}{12}(\beta-\alpha)^2, E(X_{(1)}) = \frac{1}{4}(3\alpha+\beta), E(X_{(3)}) = \frac{1}{4}(\alpha + 3\beta), Var(X_{(1)}) = Var(X_{(3)}) = \frac{3}{80}(\beta - \alpha)^2$, and $Cov(X_{(1)}, X_{(3)}) = \frac{1}{80}(\beta - \alpha)^2$)

2. Let $X_1, \ldots, X_n$ be i.i.d. from the $N(\mu, \sigma^2)$ distribution with unknown $\mu \in R$ and $\sigma > 0$. Consider estimating $\theta = \mu^2$.

   (a) Calculate the bias of $T_{1,n} = \bar{X}^2$ as an extimate of θ.

   (b) Obtain an $n^{1/2}$ order asymptotic distribution for $T_{1,n}$.

   (c) Find an unbiased estimator of θ based on the complete and sufficient statistics $(\bar{X}, S^2)$. Denote it by $T_{2,n}$.

   (d) Find the asymptotic relative efficiency of $T_{1,n}$ w.r.t. $T_{2,n}$. (Hint: $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$, $S^2 \sim \sigma^2\chi^2_{n-1}$, and they are independent of each other. Moreover, $E((N(\mu, \sigma^2)^4) = 3\sigma^4 + 6\sigma^2\mu^2 + \mu^4$ and $Var(\chi^2_d) = 2d$)

3. Let $X_1, \ldots, X_n$ be a sample from the gamma distribution $\Gamma(a, b)$ with density

$$\frac{1}{\Gamma(a)b^a} x^{a-1} e^{-x/b}, \quad x > 0, \ a, b > 0\,.$$

   Show that there exist a UMP test for testing $H_0 : b \leq b_0$ against $H_a : b > b_0$ when a is known. Please give the form of the reject region.

1. Let $X_1,\ldots,X_n$ be a random sample from a $N(\theta,\sigma^2)$ population, where σ^2 is unknown. (22%)
   a. Show that the interval $\theta \le \bar{x} + t_{n-1,\alpha}\frac{s}{\sqrt{n}}$ can be derived by inverting the acceptance region of an LRT (likelihood ratio test).
   b. Show that the interval in part (a) is a UMA (uniformly most accurate) unbiased interval.

2. Let $X_1,\ldots,X_n$ be iid Poisson(λ). (20%)
   a. Find a UMP test of $H_0:\lambda \le \lambda_0$ versus $H_1:\lambda > \lambda_0$.
   b. Consider the specific case $H_0:\lambda \le 1$ versus $H_1:\lambda > 1$. Use the Central Limit Theorem to determined the sample size n so a UMP test satisfies $P($reject $H_0 \mid \lambda = 1)$=0.5 and $P($reject $H_0 \mid \lambda = 2)$=0.9.

3. Let $X_1,\ldots,X_n$ be independent random variables, each having the Poisson distribution with parameter λ: $P(X_i = k) = \frac{\lambda^k e^{-\lambda}}{k!}$. (25%)
   a. Show that $T = \sum_{i=1}^{n} X_i$ is sufficient for λ.
   b. Suppose that we are interested in estimating $P(X \le 1) = e^{-\lambda} + \lambda e^{-\lambda}$. One estimator is $Y_1 = n^{-1}\sum_{i=1}^{n} I(X_i \le 1)$. What are the mean and variance of Y_1.
   c. Find $E(Y_1 \mid T)$.
   d. Let $Y_2 = E(Y_1 \mid T)$. Without making further calculations, how can you be sure that $E(Y_2) = e^{-\lambda} + \lambda e^{-\lambda}$ and $\operatorname{Var} Y_2 \le \operatorname{Var} Y_1$?
   e. What is the maximum likelihood estimator of $e^{-\lambda} + \lambda e^{-\lambda}$? Can you assert that Y_2 has smaller variance than this estimator?

4. Consider the two-way classification without interactions,

$$y_{i,j,k} = \mu + \alpha_i + \beta_j + \varepsilon_{i,j,k}, \quad k = 1,\ldots,n_{i,j}$$

   and suppose that both factors have 3 levels. (20%)
   a. Show how matrices **y**, **X**, **b** and **e** can be set up so that these equations can be put in the standard form **y** = **Xb** + **e**.

b. Demonstrate that the **X** matrix is not of full column rank.

c. Show that $\mu + \alpha_i + \beta_j$ is estimable if and only if $n_{i,j} > 0$.

d. Show that if:

$n_{1,2} = n_{1,3} = n_{2,1} = n_{3,1} = 0$, then $\alpha_1 - \alpha_2$ is not estimable.

5. Let $Y_1 < Y_2 < Y_3 < Y_4 < Y_5$ be the order statistics of a random sample of size 5 form the uniform distribution having p.d.f. $f(x \mid \theta) = \theta^{-1}$, $0 < x < \theta$ and $0 < \theta < \infty$. (13%)

   a. Show that $6Y_1$ is an unbiased estimator of θ.
   b. Is $E(6Y_1 \mid Y_3)$ UMVU? If not, find the UMVU estimator of θ?