| 考試科目 Course | 數理統計 | 開課系級 Dept, & Class | 統計所博士班 | 日期 Date, Period | 月 日 第 節 | 試題編號 CourseNo. | 第3卷 |
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以下4題，每題25分.

1. Let $X_1, \cdots, X_n$ be i.i.d. according to a distribution from a family $P_\theta$. Find minimal sufficient statistics for the following cases:

   (i) $P_\theta = \{ U(\theta, 2\theta),\ 0 < \theta < \infty \}$

   (ii) $P_\theta = \{ U(\theta - \frac{1}{2}, \theta + \frac{1}{2}),\ -\infty < \theta < \infty \}$,

   where $U(a,b)$ denotes the uniform distribution on $[a, b]$.

2. Let $X = (X_1, \cdots, X_n)$ be a sample from the uniform distribution on $(0, \theta)$.

   (i) Show that for testing $H: \theta \le \theta_0$ against $K: \theta > \theta_0$, any test $\phi$ is UMP (uniformly most powerful) at level $\alpha$ for which $E_{\theta_0}\phi(X) = \alpha$, $E_\theta \phi(X) \le \alpha$ for $\theta \le \theta_0$, and $\phi(x) = 1$ when $\max(x_1, \cdots, x_n) > \theta_0$.

   (ii) Determine the UMP tests for testing $\theta = \theta_0$ against $\theta < \theta_0$.

   [Hint: the density $P_\theta(x)$ has MLR in $X_{(n)} \equiv \max(x_1, \cdots, x_n)$]

3. Let $\theta$ have prior $\pi(d\theta)$, $X|\theta$ have density $f(x|\theta)$. The mean squared error of an estimator $T(x)$ is defined by $E(T-\theta)^2 = \iint (T(x) - \theta)^2 f(x|\theta)\, dx\, \pi(d\theta)$

   (i) Show that the best <u>linear</u> estimate of $\theta$ is

   $$\tilde{T}(x) = E_\pi \theta + \frac{Cov(\theta, X)}{Var(X)} (x - EX)$$

   and find its MSE.

   (ii) Suppose now that $E(X|\theta) = \theta$, $Var(X|\theta) = a + b\theta + c\theta^2$, $a, b, c$ known. Find expressions for $\tilde{T}(x)$ and MSE in terms of $x$, $EX$, $Var X$ (where the latter expectations are taken w.r.t. to the marginal density of $X$, and the term for MSE is allowed to also contain $E(X-\theta)^2$).

4. (i) Toss a $\theta$-coin till $r$ 1's occur. Let $N$ denote the number of tosses needed. Find the probability density function of $N$.

   (ii) [illegible] Let $S$ denote the number of 1's.