統計所博士班學科考試

(榮譽第一)

國立政治大學七十九學年度第　學期　考試試題

NATIONAL CHENGCHI UNIVERSITY EXAMINATION FORM

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考試科目 Course	數理統計	開課系級 Dept, & Class		日期 Date, Period	月　日 第　節	試題編號 Course No.	

1. Let $X_1, \cdots, X_n$ be n independent observations of a random variable X with probability function

(25)
$$P(X=x) = \frac{a_x \theta^x}{f(\theta)} \quad x=0,1,2,\cdots \quad 0<\theta<1$$

(a) Show that the distribution of $T = X_1+X_2+\cdots+X_n$ is of the form

$$P(T=t) = \frac{b_t \theta^t}{g(\theta)} \quad t=0,1,2,\cdots$$

and determine $g(\theta)$.

(b) Compute the conditional probability function of $X_1, \cdots, X_n$ given $T=t$. Use this probability function to show that T is a sufficient statistic

(c) Show that T is also complete

(d) Is T minimal sufficient? why or why not?

(e) Let r be a given positive integer. Find minimum variance unbiased estimator of θ^r.

2. Let $X_1, X_2, \dots, X_n$ be i.i.d. with $E(X_i)=\theta$, $var(X_i)=1$, $E(X_i-\theta)^4=\mu_4$, and consider the unbiased estimators

(10)
$$\delta_{1n} = \frac{1}{n}\sum X_i^2 - 1 \quad \text{and}$$
$$\delta_{2n} = \bar{X}_n^2 - \frac{1}{n}$$

of θ^2

(i) Determine the ARE $e_{2,1}$ of δ_{2n} with respect to δ_{1n}

(ii) Show that $e_{2,1} \geq 1$ if the X_i are symmetric about θ.

3. Let X and Y be independent r.v's with geometric distribution

$$p(X=x, Y=y) = (1-\theta_1)(1-\theta_2)\,\theta_1^{x}\,\theta_2^{y}$$

(15)
$$x=0,1,2,\dots$$
$$y=0,1,2,\dots$$

where $0<\theta_i<1$, $i=1,2$. Find a U.M.P. unbiased test of size $\alpha=.20$ for testing

(a) $H_0: \theta_1 \leq \theta_2$ against $H_1: \theta_1 > \theta_2$

(b) $H_0: \theta_1 = \theta_2$ against $H_1: \theta_1 \neq \theta_2$

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Estimation parts:

1. Consider a random variable X with density function $f(x,\theta)$ depending on a real parameter θ, and denote $l(x,\theta)=\frac{\partial}{\partial\theta}\log f(x,\theta)$ (assumed to exist). For fixed x, consider a function $h(x,\theta)$. We shall refer to h as an estimating function since a root of it, say $\hat{\theta}(x)$ where $h(x,\hat{\theta}(x))=0$, may be used to estimate θ. We shall say that h is unbiased estimating function if $\mathcal{E}_\theta\{h(X,\theta)\}=0$, $\forall\theta$. We shall say that h is standardized if $\mathcal{E}_\theta\{\frac{\partial}{\partial\theta}h(X,\theta)\}$ exists and $=1$ for all θ. (a) By suitable choice of $c(\theta)$, show that (under regularity assumptions) $h^*(x,\theta)=l(x,\theta)/c(\theta)$ is a standardized unbiased estimation function (SUEF). (b) Show that if h is any SUEF, then $Var_\theta\{h^*(X,\theta)\}\le Var_\theta\{h(X,\theta)\}$ for all θ. (15%)

2. Explain carefully, with reference to at least one example, the use of the two principles of sufficiency and invariance in the minimax estimating problem.

The independent samples $x_1,\dots,x_n$ come from a normal population of unknown mean μ and unknown standard deviation σ. It is desired to estimate the percentile $\mu+c\sigma$, where c is a known constant. Deduce from the principles of sufficiency and invariance that under general conditions on the loss function one need consider only estimates of the form $\bar{x}+\alpha s$, where $\bar{x}=n^{-1}\sum_{i=1}^{n}x_i$ and $s^2=(n-1)^{-1}\sum_{i=1}^{n}(x_i-\bar{x})^2$ and α is a constant. If the loss function is

$$\left(\frac{\bar{x}+\alpha s-\mu-c\sigma}{\sigma}\right)^2$$

deduce that a minimax invariant estimate of the percentile $\mu+c\sigma$ is $\bar{x}+c\left[\Gamma\left(\frac{n}{2}\right)\Big/2^{\frac{1}{2}}\Gamma\left(\frac{n+1}{2}\right)\right]s$ (20%)

命題教師: (簽章) 年 月 日

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Hypothesis testing part:

1. Let $\Theta \subset \mathbb{R}$, X_i, $i=1,\cdots,n$ are the i.i.d random variable with density $f_\theta(x_i)$. Assume the densities $\{f_\theta(\cdot), \theta \in \Theta\}$ form a MLR family (in the sense that if $\theta < \theta'$, $f_{\theta'}(y)/f_\theta(y)$ is a monotone increasing function of y). For the problem of testing the null hypothesis

$$H_0: \theta = \theta_0$$

against the alternative

$$H_1: \theta > \theta_0$$

where θ_0 is known, does there exist any unbiased test for it? Prove or disprove it. (15%)

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一、令 $\underline{Y} \sim N_n(\underline{\theta}, \sigma^2 I_n)$ 及 P 為 rank r 之 $n \times n$ 对称矩陣. 試証

$$Q = (\underline{Y}-\underline{\theta})' P (\underline{Y}-\underline{\theta}) / \sigma^2 \sim \chi^2_r \text{ if and only if } P^2 = P.$$

二、設 $E[\underline{Y}] = X\underline{\beta}$，$X$ 為 rank p 之 $n \times p$ 矩陣，且 covariance matrix $D[\underline{Y}] = \sigma^2 I_n$，則

$$S^2 = \frac{1}{n-p}(\underline{Y} - X\hat{\underline{\beta}})'(\underline{Y} - X\hat{\underline{\beta}})$$

為 σ^2 之不偏估計量，其中 $\hat{\underline{\beta}} = (X'X)^{-1}X'\underline{Y}$.

三、已知线性模型 $Y_i = \beta_0 + \beta_1 x_{i1} + \cdots + \beta_{p-1} x_{i,p-1} + \varepsilon_i$，$i = 1, 2, \cdots, n$；$\varepsilon_i$，$1 \le i \le n$，為互相獨立 $N(0, \sigma^2)$. 試求出檢定统計量檢定假設 $H: \beta_1 = \beta_2 = \cdots = \beta_{p-1} = 0$.