國立政治大學 102 學年度博士班資格考考試命題紙

考試科目	數理統計專題	卷別	第一卷	考試日期	103年2月10日 星期一

1. Suppose that the observations are known to have distribution uniform on $[-\theta,\theta]$.
   (a) Find the method-of-moment estimator of θ, $\tilde{\theta}_m$.
   (b) Find the maximum likelihood estimator of θ, $\hat{\theta}_l$
   (c) $\mathrm{Var}(\tilde{\theta}_m)$ and $\mathrm{E}(\tilde{\theta}_m)$.
   (d) $\mathrm{Var}(\hat{\theta}_l)$ and $\mathrm{E}(\hat{\theta}_l)$.
   (e) Find the UMVUE of θ. (30%)

2. Consider the one-factor random effect model, $Y_{ij} = \mu + \alpha_i + \varepsilon_{ij}$,
   where $\alpha_i \sim N(0,\sigma_\alpha^2)$ iid, $i = 1,\cdots,a$ treatments.
   $\varepsilon_{ij} \sim N(0,\sigma_\varepsilon^2)$ iid, $j = 1,\cdots,n$ units per treatment, and
   α_i and ε_{ij} are independent of each other.
   (a) Prove that MSA and MSE are independent under both the null $H_0: \sigma_\alpha^2 = 0$ and alternative hypotheses $H_1: \sigma_\alpha^2 > 0$.
   (b) Derive the distribution of $F = \frac{MSA}{MSE}$ under both H_0 and H_1.

   (30%)

3. Let $X_1,\cdots,X_n$ denote a random sample from a population with mean μ and variance $\sigma^2 > 0$. Let $S^2 = \sum_{i=1}^{n}(X_i - \bar{X})^2/(n-1)$ denote the usual sample variance.
   (a) Using a famous inequality, it can easily be shown that $E(S) < \sigma$.
   (b) Is S an unbiased estimator of σ?
   (c) Now assume that $X_1,\cdots,X_n$ denote a random sample from a normal population. Find an expression for $E(S)$ in terms of gamma function and other terms.
   (d) Still considering the normal model of part (c), determine the constant k such that kS is an unbiased estimator of σ.
   (40%)

考試科目	數理統計專題	卷別	第二卷	考試日期	103年2月10日 星期一

1. Suppose $X_1, X_2, \ldots, X_n$ are IID $U(-\theta, \theta)$ ($f(x|\theta) = \frac{1}{2\theta}\mathbf{1}_{-\theta \le x \le \theta}$), where $\theta \in \Theta = (0, \infty)$.

   (a) (10pts). Please find the MLE of θ, called $\hat{\theta}$.

   (b) (10pts). Please find c_n such that $\tilde{\theta} = c_n\hat{\theta}$ is unbiased.

   (c) (20pts). Please find a_n and b_n such that $a_n(\tilde{\theta} - b_n)$ has a nondegenerate limiting distribution, and give the limiting distribution.

2. Suppose X has Cauchy distribution $Cau(0, \theta)$ ($f(x|\theta) = \frac{1}{\pi\theta(1+(\frac{x}{\theta})^2)}$), where $\theta \in \Theta = (0, \infty)$.

   (a) (16pts). Please show that the family of distribution for X has no monotone likelihood ratio (MLR) property. However, the family of distribution for $Y = |X|$ has MLR.

   (b) (14pts). Please find the uniformly most powerful (UMP) level α test for H$_0$: $\theta \le \theta_0$ versus H$_1$: $\theta > \theta_0$, where θ_0 is a positive constant.

3. A large shipment of parts is received and it is desired to estimate the detective proportion θ. Consider two experimental settings. In first experiment, a sample of n parts is tested and the number of defective parts, X, then has a $Bin(n, \theta)$ distribution ($f(x|\theta) = C^n_x\theta^x(1-\theta)^{(n-x)}$). For second experiment, we are sampling parts until m of defective parts has occurred. Then the number of parts we sampled, Y, has a $NB(m, \theta)$ distribution ($g(y|\theta) = C^{y-1}_{m-1}\theta^m(1-\theta)^{y-m}$). From past shipments, suppose it is known that θ has a $Beta(\alpha, \beta)$ distribution ($\pi(\theta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}\mathbf{1}_{0\le\theta\le1}$).

   (a) (10pts). Consider the first experiment. Please find the posterior distribution of θ given X, $\pi(\theta|X)$, and the Bayesian estimate, $\hat{\theta}$, under the squared-error loss.

   (b) (10pts). Consider the second experiment. Please find the posterior distribution of θ given Y, $\pi(\theta|Y)$, and the Bayesian estimate, $\tilde{\theta}$, under the squared-error loss.

   (c) (10pts). Explain whether or not the two experimental settings convey the same information by finding the relationship between $\hat{\theta}$ in (a) and $\tilde{\theta}$ in (b).

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考試科目	數理統計專題	卷　別	第三卷	考試日期	103年2月10日 星期一

1. (10%) Let $X_1, \dots, X_n$ be i.i.d. as $N(\theta, 1)$, and let $g(\theta) = 0$ if $\theta \neq 0$ and $g(0) = 1$. Find a consistent estimator of $g(\theta)$.

2. Let $X_1, \dots, X_n$ be i.i.d. with $EX_i = \mu$, $Var(X_i) = 1$, and $E(X_i^4) < \infty$. Let $T_{1n} = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - 1$ and $T_{2n} = \bar{X}^2 - \frac{1}{n}$ be estimators of μ^2.
(10%) (a) Find the asymptotic relative efficiency $e_{T_{1n}, T_{2n}}$ of T_{1n} w.r.t. T_{2n}.
(6%) (b) Show that $e_{T_{1n}, T_{2n}} \leq 1$ if the distribution of $X_i - \mu$ is symmetric about 0 and $\mu \neq 0$.

3. Let X take on the values 1 and 0 with probability p and $1-p$, respectively, and assume that $\frac{1}{4} < p < \frac{3}{4}$. Consider the problem of estimating p with loss function $L(p, d) = 1$ if $|d - p| \geq \frac{1}{4}$, and 0 otherwise. Let δ be the randomized estimator which is Y_0 or Y_1 when $X = 0$ or 1 where Y_0 and Y_1 are distributed as $U(-\frac{1}{2}, \frac{1}{2})$ and $U(\frac{1}{2}, \frac{3}{2})$, respectively.
(6%) (a) Show that δ is unbiased.
(10%) (b) Compare the risk function of δ with that of X.

4. Let X be a random variable having the Poisson distribution $P(\theta)$ with an unknown $\theta > 0$.
(10%) (a) Show that $\sup_\theta E_\theta(T(X) - \theta)^2 = \infty$ for any estimator $T(X)$.
(8%) (b) Let $\mathcal{F} = \{aX + b : a, b \in R\}$. Show that 0 is a $\mathcal{F}$-admissible estimator of θ under the squared error loss.
(5%) (c) Show that there exists no unbiased estimator for $\frac{1}{\theta}$.

5. Let $X_1, \dots, X_n$ be i.i.d. random variables from the uniform distribution $U(0, \theta)$, where $\theta > 0$.
(10%) (a) Derive a UMP test of size α for testing H_0: $\theta \leq \theta_0$ versus H_1: $\theta > \theta_0$.
(10%) (b) Derive a UMP test of size α for testing H_0: $\theta = \theta_0$ versus H_1: $\theta \neq \theta_0$.

6. (15%) For each $\theta_0 \in \Theta$, let $A(\theta_0)$ be the acceptance region of a level-α test for testing $H_0(\theta_0) : \theta = \theta_0$, and for each sample point $\mathrm{x} = (x_1, \dots, x_n)$, let $S(\mathrm{x})$ denote the set of parameter values $S(\mathrm{x}) = \{\theta : \mathrm{x} \in A(\theta), \theta \in \Theta\}$. Then $S(\mathrm{x})$ is a family of confidence sets for θ at confidence level $1-\alpha$. If, moreover, $A(\theta_0)$ is UMP for testing $H_0(\theta_0)$ at level α against the alternative $H_1(\theta_0)$, then for each $\theta_0 \in \Theta$, $S(\mathrm{X})$ minimizes the probability $P_\theta\{\theta_0 \in S(\mathrm{X})\}$ for all $\theta \in H_1(\theta_0)$ among all level-$(1-\alpha)$ families of confidence sets for θ. That is, $S(\mathrm{X})$ is UMA.