國立政治大學 101 學年度博士班資格考考試命題紙

考試科目	數理統計專題	卷別	第一卷	考試日期	101年9月10日 星期一

1. (10 points) Suppose that X is a random variable with CDF F_X. Show that $E(X) = -\int_{-\infty}^{0} F_X(x)dx$ if $X \le 0$.

2. (10 points) Suppose that the distribution of a random vector X belongs to a parametric family $\{P_\theta : \theta \in \Theta\}$. Suppose that X can be observed. Consider the problem of estimating θ based on X. Let $\mathcal{U}$ be the set of all unbiased estimator of 0. That is,

$$\mathcal{U} = \{U(X) : E_\theta(U(X)) = 0 \text{ for all } \theta \in \Theta\}.$$

Suppose that $\delta(X)$ is an unbiased estimator of θ. Show that $\delta(X)$ is an UMVUE of θ if and only if for any $U(X) \in \mathcal{U}$, $Cov(\delta(X), U(X)) = 0$.

3. Suppose that $(X_1, \ldots, X_n)$ is a random sample from the uniform distribution on $(0, \theta)$, where $\theta > 0$. Let $Pa(b, \lambda)$ denote the Pareto distribution with parameters b and λ. The probability density function for $Pa(b, \lambda)$ is given as follows:

$$f(x) = \lambda b^\lambda x^{-\lambda-1} I_{(b,\infty)}(x),$$

where $I_{(b,\infty)}$ denotes the indicator function on (b, ∞), and the parameters b and λ are positive. Consider the problem of estimating θ under the loss function

$$L(\theta, a) = \frac{(a-\theta)^2}{\theta^2}.$$

(a) (10 points) Show that the Bayes estimator for θ under the loss L with respect to the prior distribution $Pa(b, \lambda)$ is

$$\frac{(\lambda + n + 2)\max(b, X_{(n)})}{\lambda + n + 1},$$

where $X_{(n)}$ denotes the n-th order statistic of $(X_1, \ldots, X_n)$.

(b) (10 points) Let $Y = \max(b, X_{(n)})$. Show that

$$E(Y) = \begin{cases} b & \text{if } b \ge \theta; \\ \frac{n\theta}{n+1} + \frac{b^{n+1}}{(n+1)\theta^n} & \text{if } b < \theta, \end{cases}$$

and

$$E(Y^2) = \begin{cases} b^2 & \text{if } b \ge \theta; \\ \frac{n\theta^2}{n+2} + \frac{2b^{n+2}}{(n+2)\theta^n} & \text{if } b < \theta. \end{cases}$$

(c) (10 points) Show that $(n+2)X_{(n)}/(n+1)$ is a minimax estimator of θ under the loss L.

4. Suppose that X_1 and X_2 are IID Bernoulli(θ) and $X = X_1 + X_2$. Suppose that $\theta \in \{0.25, 0.5\}$ and only X is observed. Consider the problem of estimating θ based on X under squared error loss with action space $\{0.25, 0.5\}$. That is, the loss for estimating θ using a is $(a-\theta)^2$, where $a \in \{0.25, 0.5\}$.

考試科目	数理統計專題	卷別	第一卷	考試日期	101年9月10日 星期一

(a) (10 points) Consider the prior that puts probability p on 0.25 and probability $1-p$ on 0.5. Find the Bayes rule for estimating θ under squared error loss with action space $\{0.25, 0.5\}$ with respect to this prior.

(b) (10 points) Let $R_p(\theta)$ denote the risk function for the Bayes rule in Part (a). Let $f(p) = R_p(0.25)/R_p(0.5)$. Give a formula for computing $f(p)$ but do not simplify the expression.

5. Suppose that we have a random sample $(X_1, \ldots, X_n)$ from a distribution with a Lebesgue probability density function f, which is given by

$$f(x) = \frac{\beta^\lambda}{\Gamma(\lambda)} \exp(\lambda x - \beta e^x) \text{ for all } x \in (-\infty, \infty),$$

where β and λ are positive parameters and $\Gamma(\cdot)$ is the Gamma function, which is defined by $\Gamma(a) = \int_0^\infty t^{a-1} e^{-t} dt$ for $a > 0$. Suppose that α is a constant in $(0, 1)$.

(a) (10 points) Suppose that $\lambda = 1$. Find a statistic U that is complete and sufficient for β.

(b) (10 points) Suppose that $\lambda = 1$. Let $Y_i = e^{X_i}$ for $i = 1, \ldots, n$, $\bar{Y} = n^{-1} \sum_{i=1}^n Y_i$ and $V = \prod_{i=1}^n (Y_i/\bar{Y})$. Show that V is independent of the statistic U in part (a).

(c) (10 points) Find a UMPU level α test for the problem

$$H_0 : \lambda \leq 1 \text{ versus } H_1 : \lambda > 1$$

Also, express the rejection region of the UMPU test using the statistic V in part (b), if possible.

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考試科目	數理統計專題	卷別	第二卷	考試日期	101年9月10日 星期一

1. (20 points) (a) Given real-valued functions f_n, $n \geq 1$ and f for which $f_n(x) \to f(x) \forall x \in \Re$, but $\lim_{n\to\infty} \int f_n(x)dx \neq \int f(x)dx$.
   (b) What is Lebesgue Dominated Convergence Theorem?

2. (30 points) Let $X_1, ..., X_n$ be iid from $N(0, \tau)$ population.
   (a) Give the meaning of "conjugate prior."
   (b) Find the conjugate prior of τ.
   (c) Now choose a prior for τ from the conjugate family and then find the posterior distribution of τ.

3. (20 points) Consider a linear model:

$$y_i = x_i^T \beta + e_i, \quad i = 1, ..., n,$$

   where $x_i \in \Re^p$ are fixed values, e_i are iid $N(0,1)$ variates, and $\beta \in \Re^p$ is an unknown parameter. The above model can be written in matrix form:

$$y = X\beta + e,$$

   where X is an $n \times p$ matrix. Let $\hat{\beta}$ be the MLE of the β
   (a) Derive $p(\hat{\beta})$, the distribution of $\hat{\beta}$.
   (b) From Bayesian perspective, β is considered random. Now suppose that β has a flat prior $p(\beta) \propto 1$. Obtain $p(\beta|y)$, the posterior distribution of β given data.

4. (30 points) Let X_n and X be random variables.
   (a) (5 points) What does it mean that X_n converge to X with probability 1 (w.p.1)?
   (b) (5 points) What does it mean that X_n converge to X in probability?
   (c) (5 points) Suppose that X_n's are iid and $EX_n = \mu$. Use these notations to state the strong law of large numbers.
   (d) (10 points) Argue that $X_n \to X$ w.p.1 if and only if $P(|X_n - X| \geq \epsilon \text{ i.o.}) = 0$ for each $\epsilon > 0$, where "i.o." mean infinite often.
   (e) (5 points) Use (d) to show that convergence w.p.1 implies convergence in probability.

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考試科目	數理統計專題	卷別	第三卷	考試日期	101年9月10日 星期一

1. Let $\{X_n, n \geq 1\}$ be a sequence of random variables defined on a probability space with $P(X_n = -n) = P(X_n = n) = \frac{1}{2n^2}$ and $P(X_n = 0) = 1 - \frac{1}{n^2}$.
(a) (7%) Does $\{X_n\}$ converge in distribution? If so, to what? Why?
(b) (8%) Does $\{X_n\}$ converge almost surely? If so, to what? Why?

2. (15%) Let X have distribution P_θ, $\theta \in \Omega$, let δ be an estimator in $\triangle = \{\delta : E_\theta \delta^2 < \infty, \ \forall \theta\}$, and let $\mathcal{U}$ denote the set of all unbiased estimators of zero which are in $\triangle$. Then, a necessary and sufficient condition for δ to be a UMVU estimator of its expectation $g(\theta)$ is that $E_\theta(\delta U) = 0$ for all $U \in \mathcal{U}$ and all $\theta \in \Omega$.

3. Let X be the binomial distribution $b(n, p)$.
(10%) (a) Find the unique minimax estimator of p for the loss function $\frac{(d-p)^2}{p(1-p)}$.
(15%) (b) Determine which estimators $a\frac{X}{n} + b$ are admissible for estimating p for squared error loss.

4. (15%) Let the distribution of X is given by

x	0	1	2	3
$P_\theta\{X = x\}$	θ	2θ	$0.9 - 2\theta$	$0.1 - \theta$

where $0 < \theta < 0.1$. For testing $H : \theta = 0.05$ against $K : \theta > 0.05$ at level $\alpha = 0.05$, determine which of the following tests (if any) is UMP:
(i) $\phi(0) = 1$, $\phi(1) = \phi(2) = \phi(3) = 0$;
(ii) $\phi(1) = 0.5$, $\phi(0) = \phi(2) = \phi(3) = 0$;
(iii) $\phi(3) = 1$, $\phi(0) = \phi(1) = \phi(2) = 0$.

5. Let $X_1, \ldots, X_n$ be random sample from the normal distribution $N(\mu, \sigma^2)$, where μ and σ are unknown parameters. Consider the problem of testing H: $\sigma \geq \sigma_0$ against K: $\sigma < \sigma_0$.
(10%) (a) Find the UMP unbiased level-α test.
(10%) (b) Find the UMP invariant level-α test under the group of transformations $g_c(X_i) = X_i + c$, $c \in R$.

6. (10%) Let $X_1, \ldots, X_n$ be a random sample from a Pareto pdf $f_\theta(x) = \frac{\theta}{x^2}$, $x > \theta$, and $= 0$ for $x \leq \theta$, where $\theta > 0$. Show that the shortest-length $1 - \alpha$-level confidence interval for θ based on $X_{(1)}$ is $(X_{(1)}\alpha^{\frac{1}{n}}, X_{(1)})$.