考試科目	線性模式專題	卷別	第一卷	考試日期	99 年 9 月 6 日 星期一

1. Let Y be a random vector in R^n having a multivariate normal distribution with mean vector 0 and covariance matrix D, a nonsingular positive definite $n \times n$ matrix. Denote the eigenvalues of D by $\lambda_1, \ldots, \lambda_n$ and the corresponding orthogonal eigenvectors by $v_1, \ldots, v_n$. Suppose v_i are unit vectors and denote P_i the $n \times n$ projection matrix to v_i for $i = 1, \ldots, n$.

   (a) (12pts). Please show that $D = \Sigma_{i=1}^{n} \lambda_i P_i$.

   (b) (16pts). Please show that the random variable $Q = Y'Y$ is a linear combination with coefficients $\lambda_1, \ldots, \lambda_n$ of independent χ_1^2 random variables. (Hint: $Z'P_1Z, \ldots, Z'P_nZ$ are independently identically distributed χ_1^2 for standardized normal distribution Z.)

   (c) (4pts). Find $E(Q)$.

   (d) (4pts). Find $Var(Q)$.

2. Suppose that $Y_{ij} = \mu + \alpha_i + \beta_j + \epsilon_{ij}$ and $E(\epsilon_{ij}) = 0$ for $i = 1, 2$ and $j = 1, 2, 3$.

   (a) (6pts). Write the model in vector form.

   (b) (10pts). Find conditions on c_1, c_2, c_3, such that $\eta = c_1\beta_1 + c_2\beta_2 + c_3\beta_3$ is estimable; namely, there exists a linear unbiased estimator for η.

   (c) (2pts). Show that $\beta_2 - \beta_1$ is estimable.

   (d) (8pts). Give two unbiased estimators of $\beta_2 - \beta_1$.

考試科目	線性模式專題	卷別	第一卷	考試日期	99 年 9 月 6 日 星期一

3. Consider the multiple linear regression model: $Y = X\beta + \epsilon$, where $Y = \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix}$, $X = \begin{pmatrix} x_1, \cdots x_k \end{pmatrix}$, $x_1, \ldots, x_k$ are vectors of constant in R^n, $\beta = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_k \end{pmatrix}$, and $\epsilon = \begin{pmatrix} \epsilon_1 \\ \vdots \\ \epsilon_n \end{pmatrix}$. Suppose that X has rank k and $\epsilon_1, \ldots, \epsilon_n$ are independent with the same mean 0 and variance σ^2.

   (a) (4pts). Please write down the best linear unbiased estimator (BLUE) of β in terms of X and Y. (you are not required to verify your answer for this question).

   (b) (12pts). Denote the BLUE in (a) by $\hat{\beta} = \begin{pmatrix} \hat{\beta}_1 \\ \vdots \\ \hat{\beta}_k \end{pmatrix}$. Let $x_k^{\perp} = x_k - P(x_k | x_1, \ldots, x_{k-1})$, the part of x_k which is orthogonal to the other x_j. Please derive a formula for $\hat{\beta}_k$ in terms of $x_k^{\perp}$.

   (c) (8pts). Denote the regression sum of squares under the general model by SSR_k and under the null hypothesis $\beta_k = 0$ by SSR_{k-1}. Please derive a formula for SSR_k-SSR_{k-1} in terms of $x_k^{\perp}$ and Y.

   (d) (6pts). Consider the following pairs (x_{ijl}, y_{ijl}) observed for $i = 1, 2$; $j = 1, 2$; $l = 1, 2$:

	B_1	B_2
A_1	(10, 25) (14, 23)	(7, 8) (5, 12)
A_2	(1, 20) (7, 12)	(0, 7) (4, 5)

   For the analysis of covariance model $Y_{ijl} = u_{ij} + \gamma x_{ijl} + \epsilon_{ijl}$, consider $Y = \begin{pmatrix} Y_{111} \\ Y_{112} \\ Y_{121} \\ Y_{122} \\ Y_{211} \\ Y_{212} \\ Y_{221} \\ Y_{222} \end{pmatrix}$ and $\beta = \begin{pmatrix} \mu_{11} \\ \mu_{12} \\ \mu_{21} \\ \mu_{22} \\ \gamma \end{pmatrix}$. Please give Y and X for this data set.

   (e) (8pts). Please compute the BLUE for γ.

國立政治大學 99 學年度博士班資格考考試命題紙

考試科目	線性模式專題	卷別	第二卷	考試日期	99 年 9 月 6 日 星期一

1. (20%) Suppose $Y \sim N(\theta, \Sigma)$, where Σ is a nonsingular, positive definite $n \times n$ matrix. Let $A_{n\times n}$ be a nonnegative definite matrix, derive the distribution of $Y'AY$. In particular, discuss the case when $\Sigma = \sigma^2 I_n, A = P/\sigma^2$, for P a projection matrix onto a d-dimentional subspace V of R_n.

2. (50%) Let $Y = \sum_1^k \beta_j x_j + \epsilon$ and $\epsilon \sim N(0, \sigma^2 I_n)$. Assume the design matrix X has full column rank.

   (a) (15%) Derive the least squares estimator $\hat{\beta}$ and show that it is BLUE.

   (b) (10%) Derive the sampling distribution of $\hat{\beta}$.

   (c) (15%) Derive a α-level statistical procedure to test $H_0 : A\beta = 0$ for a given $A_{q\times k}$, a matrix of rank q.

   (d) (10%) If in particular, X is not of full column rank, show that the parameter $\eta = c'\beta$ is estimable if and only if c lies in the row space of X.

3. (20%)Let Z_1, Z_2 and B be independent random variables with Z_1, Z_2 standard normal and B Bernoulli(0.5). Let $Y_1 = (2B-1)|Z_1|$ and $Y_2 = (2B-1)|Z_2|$.

   (a) (10%) Show that marginally Y_1, Y_2 follow the standard normal distribution.

   (b) (10%) Show that the joint distribution of (Y_1, Y_2) is not normal.

4. (10%) Assume the predictors x_1, x_2 satisfy $x_2 = x_1 + \alpha u$, where x_1, u are linearly independent in R_n. Derive the covariance matrix of $\hat{\beta}$ and see what happens as $\alpha \to 0$ or $\alpha \to \infty$.

考試科目	線性模式專題	卷別	第三卷	考試日期	99 年 9 月 6 日 星期一

1.(12%) Let $A=\begin{bmatrix} 8 & -4 & 6 \\ -4 & 8 & -6 \\ 6 & -6 & 33 \end{bmatrix}$. Find a matrix U such that $A=UU'$. Oh, by the way, one of the eigenvalues of A is 36.

2.(12%) Assume that $\mathbf{X}=\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \sim N\left(\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 & -2 & 1 \\ -2 & 5 & -2 \\ 1 & -2 & 3 \end{bmatrix}\right)$, and that $Q_1(\mathbf{X}) =$

$(X_1-2X_2)^2+(X_2-3X_3)^2+(2X_1-X_3)^2$, and $Q_2(\mathbf{X}) = 2(X_1-\bar{X})^2+(X_2-\bar{X})^2+3(X_3-\bar{X})^2$,.
Compute $E[Q_1(\mathbf{X})]$ and $E[Q_2(\mathbf{X})]$.

3.(16%) Suppose the following pairs (X_{ijk}, Y_{ijk}) are recorded for i=1, 2, j=1, 2, and k=1, 2.

	$\mathbf{B}_1$	$\mathbf{B}_2$
$\mathbf{A}_1$	(10,25)	(7,8)
	(14,23)	(5,12)
$\mathbf{A}_2$	(1,20)	(0,7)
	(7,12)	(4,5)

Complete a two-way analysis of covariance according to the following model.

$$Y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \gamma X_{ijk} + \varepsilon_{ijk}.$$

The analysis should include a number of F-tests and conclusions.

4.(12%) The effect of anesthetics on dogs are studied using a design described as follows. Five dogs were initially given the drug pentobarbital to put to sleep. Each dog was then administered carbon dioxide (CO_2) at each of three different pressure levels (low, medium, and high). The response, time between heartbeats of each dog, were recorded and shown in the table below.

	CO_2 pressure level		
Dog	**low**	**med**	**high**
1	29	42	49
2	40	50	55
3	36	44	49
4	33	46	47
5	37	43	50

Use the appropriate method to test to see whether there is a significant effect on the heartbeat rate of the dogs due to different pressure levels of CO_2. State your conclusion.

考試科目	線性模式專題	卷別	第三卷	考試日期	99 年 9 月 6 日 星期一

5.(16%) The joint probability mass function of the pair of random variables (X, Y) is tabulated as follows.

		X		
		0	1	2
Y	0	0.1	0.24	0.3
	1	0.1	0.16	0.1

Find the least squares predictor $g(\mathbf{X})=E[\mathbf{Y}|\mathbf{X}]$, and the linear least squares predictor

$h(\mathbf{X})=\hat{Y}=\mu_y+\rho\sigma_Y(\mathbf{X}-\mu_X)/\sigma_X$. Show that $g(\mathbf{X})$ and $h(\mathbf{X})$ are unbiased estimators of $E[\mathbf{Y}]$. Find

$var(g(\mathbf{X}))$, $var(\hat{Y})$, and $E[(\mathbf{Y}-g(\mathbf{X}))^2]$.

6.(12%) Assume that $\mathbf{x}_1=[1, 1, 1, 1, 1]'$, $\mathbf{x}_2=[1, 1, 1, 0, 0]'$, $\mathbf{x}_3=[1, 0, 1, 1, 1]'$, and $\mathbf{y}=[-4, -6, -3, 7, 6]'$,

and we are fitting the model $\mathbf{y}=\sum_{j=1}^{3}\beta_j x_j+\varepsilon$ to the data.

a(6%) Estimate β_1, β_2, and β_3. Compute the residual SS.

b(6%) Use α=0.05 to test $H_0: \beta_2+2\beta_3=0$ against the appropriate alternative.

7.(20%) We are modeling yields of blueberry of two different conditions with several plots within each condition. Let $\mathbf{y}=\begin{bmatrix} y_{11} & y_{21} \\ y_{12} & y_{22} \\ y_{13} & y_{23} \\ y_{14} & y_{24} \end{bmatrix}$, in which the entry y_{ij} denotes the yield of blueberry of the ith condition on the jth plot, where j=1, 2,…, n_i, i=1, 2, and $n_1=n_2=4$. Furthermore, let $\mathbf{J}_1=\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$ and $\mathbf{J}_2=\begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$ denote the condition effect indicator matrices, and the entry x_{ij} in the matrix $\mathbf{x}=\begin{bmatrix} x_{11} & x_{21} \\ x_{12} & x_{22} \\ x_{13} & x_{23} \\ x_{14} & x_{24} \end{bmatrix}$ denotes the amount of fertilization for the blueberry of ith condition on the jth plot. We are now fitting the model $\mathbf{y}=\sum_{j=1}^{2}\beta_j\mathbf{J}_j+\beta\mathbf{x}+\varepsilon$ to the yield of blueberry.

考試科目	線性模式專題	卷別	第三卷	考試日期	99 年 9 月 6 日 星期一

And now we have $\mathbf{y} = \begin{bmatrix} 90 & 110 \\ 106 & 148 \\ 126 & 122 \\ 130 & 132 \end{bmatrix}$, and $\mathbf{x} = \begin{bmatrix} 3 & 2 \\ 5 & 8 \\ 6 & 4 \\ 6 & 6 \end{bmatrix}$.

a(6%) Estimate β_1, β_2, and β.

b(9%) Compute $\hat{y}$, S^2 and the standard error of $\hat{\beta}$.

c(5%) Use α=0.05 to test $H_0: \beta_1 = \beta_2$ against the appropriate alternative.