第一卷

Qualify Exam 96.02.26

Mathematical Statistics

1.(20%) Let $X_1, \cdots, X_n$ be i.i.d. random variables from $N(\mu, \sigma^2)$ with unknown $\mu \in R$ and $\sigma^2 > 0$. Define

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

and

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2.$$

Let $Y_i = (X_i - \mu, (X_i - \mu)^2), i = 1, \cdots, n.$

(a) Find the variance-covariance matrix of Y_1, denoted by Σ.

(b) Applying the Central Limit Theorem to $Y_i's$, prove that

$$\sqrt{n}(\overline{X} - \mu, S^2 - \sigma^2) \longrightarrow_d N_2(0, \Sigma)$$

2. (20%) Let $X_1, \cdots, X_n$ be i.i.d. random variables from the Poisson distribution with unknown parameter $\theta > 0$. Consider estimation of $\vartheta = P(X_1 = 0) = e^{-\theta}$. Let T_n be an estimator of ϑ for every n. The asymptotic expectation of $(T_n - \vartheta)^2$, is defined to be an asymptotic mean squared error (amse) of T_n, denoted by $E\gamma_n^2 = amse_{T_n}(\theta)$. Suppose that there is a sequence of positive numbers $\{a_n\}$ such that $a_n \to \infty$ and

$$\lim_{t\to\infty}\lim_{n\to\infty} E[min(a_n^2 r_n^2, t] = \lim_{n\to\infty} a_n^2 E\gamma_n^2 \in (0, \infty).$$

Then $E\gamma_n^2$ is called a regular amse of T_n, denoted by $\underline{amse}_{T_n}(\theta)$. Define the empirical c.d.f.

$$F_n(t) = \frac{1}{n}\sum_{i=1}^{n} I_{(-\infty,t]}(X_i), \quad t \in R.$$

Let $T_{1n} = F_n(0)$ and $T_{2n} = e^{-\overline{X}}$.

(a) Prove that T_{1n} is unbiased and find the mean square error of T_{1n}.

(b) Prove that $\underline{amse}_{T_{1n}}(\theta) = mse_{T_{1n}}$ and $\underline{amse}_{T_{2n}}(\theta) = e^{-2\theta}\theta/n$.

(c) Find the asymptotic relative efficiency of T_{1n} w.r.t. T_{2n}.

3. (20%) Let $X_1, \cdots, X_n$ be i.i.d. random variables from $N(\mu, \sigma^2)$ with unknown $\mu \in R$ and $\sigma^2 > 0$.

(a) Prove that the UMVUE for σ^r, $r > 1-n$ is $k_{n-1,r}S^r$, where S^2 is as in Problem 1 and

$$k_{n,r} = \frac{n^{r/2}\Gamma(n/2)}{2^{r/2}\Gamma(\frac{n+r}{2})}.$$

(b) Derive the UMVUE of μ^2 and μ/σ.

4. (20%) Let $X_1, \cdots, X_n$ be i.i.d. random variables from uniform distribution $U(\theta, \theta+1)$, $\theta \in R$. Suppose $n \geq 2$.

(a) Find the joint distribution of $X_{(1)}$ and $X_{(n)}$.

(b) Show that a UMP test of size α for testing $H_0 : \theta \leq 0$ versus $H_1 : \theta > 0$ is of the form

$T(X_{(1)}, X_{(n)}) = 0$, if $X_{(1)} < 1 - \alpha^{1/n}, X_{(n)} < 1$;

$T(X_{(1)}, X_{(n)}) = 0$, otherwise.

5. (20%) Let $X_1, \cdots, X_n$ be i.i.d. with the Lebesgue p.d.f. $\theta x^{\theta-1} I_{(0,1)}(x)$, where $\theta > 0$ is unknown.

(a) Construct a confidence interval for θ with confidence coefficient $1-\alpha$, using a sufficient statistics.

(b) Discuss whether the confidence interval obtained in (a) has the shortest length within a class of confidence intervals.

(c) Discuss whether the confidence interval obtained in (a) is UMAU.

第二卷

1. (40 pts) Suppose that $X_1, \ldots, X_n$ are independent and identically distributed and $\tau = E(X_1^2) < \infty$.

   (a) Show that $n^{-1}\sum_{i=1}^n X_i^2$ is a consistent estimator for τ.

   (b) Suppose that the distribution for X_1 is $Uniform(0, \theta)$, where $\theta > 0$ is unknown. Find another consistent estimator for τ that is asymptotically more efficient than the estimator in part (a). Justify your answer.

2. (30 pts) Suppose that $X_1, \ldots, X_n$ are independent and identically distributed and the probability density function of X_1 is

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} x^{-1} e^{-(\log x - \mu)^2/2\sigma^2} I_{(0,\infty)}(x),$$

   where $\mu \in (-\infty, \infty)$ and $\sigma \in (0, \infty)$ are unknown parameters. Let $T_n = n^{-1}\sum_{i=1}^n \log X_i$ and $S_n = n^{-1}\sum_{i=1}^n (\log X_i - T_n)^2$.

   (a) Find a complete sufficient statistic for (μ, σ).

   (b) Find the UMVUE for μ.

   (c) Show that T_n and S_n are independent.

3. (30 pts) Suppose that $X_1, \ldots, X_n$ are independent and identically distributed $N(\mu, \sigma^2)$ random variables, and we are interested in testing whether $\mu = 0$. For this problem, you may use the following fact without proving it:

   Fact. Let $U = n^{-1}\sum_{i=1}^n X_i$ and $V = n^{-1}\sum_{i=1}^n X_i^2$. Then when $\mu = 0$,

   - the probability density function of $U/\sqrt{V}$ is symmetric about 0,
   - $U/\sqrt{V}$ is independent of V, and
   - the distribution of U^2/V is $Beta(1/2, (n-1)/2)$.

   (a) Suppose that μ is either 0 or 1 and $\sigma = 1$. Find a UMP test of size α for H_0: $\mu = 0$ versus H_1: $\mu = 1$.

   (b) Suppose that $\mu \in (-\infty, \infty)$ and $\sigma \in (0, \infty)$ (both are unknown). Find a UMPU test of size α for H_0: $\mu = 0$ versus H_1: $\mu \neq 0$. You may express the answer using the inverse function of the cumulative distribution function of $Beta(1/2, (n-1)/2)$.

第三卷

Mathematical Statistics: Ph.D. Qualifier Exam

February 26, 2007
(100 minutes)

1. Let $X_1, X_2, X_3, \ldots$ be i.i.d. with $\mathcal{N}(\theta, 1)$ and $Y_n = \Phi((c - \bar{X}_n)/\sqrt{1 - 1/n})$, where Φ is the standard normal distribution function, c is a constant, and $\bar{X}_n = (X_1 + X_2 + \ldots + X_n)/n$. We are interested in the limiting behaviour of Y_n. Please answer the following questions.
   (a) (6pts). Denote $T_n = c - \bar{X}_n$. Find r_n and s_n such that $r_n(T_n - s_n)$ has a nondegenerate limiting distribution, and find the distribution.
   (b) (6pts). Denote $W_n = (c - \bar{X}_n)/\sqrt{1 - 1/n}$. Find u_n and v_n such that $u_n(W_n - v_n)$ has a nondegenerate limiting distribution, and find the distribution.
   (c) (6pts). Find a_n and b_n such that $a_n(Y_n - b_n)$ has a nondegenerate limiting distribution, and find the distribution.

2. Sometimes unbiased estimators exist, but none is UMVUE (uniformly minimum variance unbiased estimator). An example follows. Suppose that P_θ says that $Y_1, Y_2, Y_3, \ldots$ are i.i.d. with Ber(θ). Set

$$X = \begin{cases} 1 & \text{if } Y_1 = 1 \\ \text{number of trials until 2nd failure} & \text{otherwise} \end{cases},$$

   and suppose that we observe only X. Please answer the following questions.
   (a) (4pts). Please give the density function of X.
   (b) (4pts). Please find an unbiased estimator of θ.
   (c) (8pts). Please prove the following statement for general situation. Let $\mathcal{U}$ be the set of all unbiased estimators of 0. Namely,

$$\mathcal{U} = \{U(X) : \mathrm{E}_\theta(U(X)) = 0 \text{ for all } \theta\}.$$

   Then $\delta(X)$ is an UMVUE of θ if and only if, for any $U(X) \in \mathcal{U}$, $\mathrm{Cov}(\delta(X), U(X)) = 0$.
   (d) (8pts). Please characterize $\mathcal{U}$ for the example given above. Namely, find conditions on $U(\cdot)$ for $U \in \mathcal{U}$ for the example given above.
   (e) (8pts). Please show that there is no UMVUE of θ for the example given above.

3. Please answer the following questions.
   (a) (10pts). Let $\{P_\theta : \theta \in \Theta\}$ be a parametric family, and let $p_\theta(x)$ be the density of a member of the family with respect to Lebesgue measure. Assume that $\partial^2 \log p_\theta(X)/\partial x \partial \theta$ exists for all x and θ. Please prove that the family has increasing MLR (monotone likelihood ratio) if and only if $\partial^2 \log p_\theta(X)/\partial x \partial \theta \geq 0$ for all x and θ.
   (b) (4pts). Please give a similar statement for case of decreasing MLR. Proof is not required.

4. Suppose that $X \sim \mathcal{N}(\theta, 1)$ and let θ have an $\mathcal{N}(0, 1)$ prior. Suppose we are interested in estimating θ and the parameter space and action space are both $(-\infty, \infty)$. Let the loss function be $L(\theta, a) = 0$ if $a \geq \theta$ and $L(\theta, a) = 1$ if $a < \theta$, where a denotes the action (estimator).

   (a) (4pts). What is the posterior of θ?

   (b) (4pts). Please compute the Bayes risk (Posterior risk) $r(\delta(X)|X)$, where $\delta(X)$ is any decision rule.

   (c) (4pts). Please show that there is no Bayes rule.

   (d) (8pts). Please show that every decision rule is inadmissible.

   (e) (12pts). Please show that if the action space is $[-\infty, \infty]$, then there is a Bayes rule and that it is the only admissible rule.

   (f) (4pts). How do you comment on the loss function? Do you take L a nice loss function in this decision problem? Explain.