87學年度第一學期

Linear Models (Comprehensive Exam)

87/10/2

1. Consider independent random variables Y_{ij}, i=1,2, j=1,2,3, with distribution

$$Y_{ij} \sim N(\alpha + \beta_i x_j, \sigma^2),\ i = 1,2,\ j = 1,2,3,$$

where

$$x_j = \begin{cases} -1 & \text{for } j = 1 \\ 0 & \text{for } j = 2 \\ 1 & \text{for } j = 3 \end{cases}$$

(a) Show that the model is a linear model, and find the estimators $\hat{\alpha}, \hat{\beta}_1$, and $\hat{\beta}_2$. Find the distributions of $\hat{\alpha}, \hat{\beta}_1$, and $\hat{\beta}_2$, and show that the estimators are mutually independent.

(b) Show that the estimator of σ^2 may be written in the form

$$\hat{\sigma}^2 = \frac{1}{3}(S_{yy} - 6\hat{\alpha}^2 - 2\hat{\beta}_1^2 - 2\hat{\beta}_2^2),$$

where $S_{yy} = \sum_{i=1}^{2}\sum_{j=1}^{3} y_{ij}^2$. Find the distribution of $\hat{\sigma}^2$.

(c) Show that the F-statistic for the hypothesis $H_0 : \beta_1 = \beta_2 = 0$ is

$$T(y) = \frac{\hat{\beta}_1^2 + \hat{\beta}_2^2}{\hat{\sigma}^2}.$$

Find the distribution of $T(Y)$ under H_0.

2. Let $\mathbf{M}$ be a $k \times k$ positive definite matrix. Let $\mathbf{A}$ be a $q \times k$ matrix of rank q. Let C be the row space of $\mathbf{A}$, and define $Q = \mathbf{A}'[\mathbf{A}\mathbf{M}^{-1}\mathbf{A}]^{-1}\mathbf{A}$.

(a) Show that for any $\mathbf{b} \in R^k$, $\sup_{\mathbf{c} \in C} \frac{(\mathbf{c}'\mathbf{b})^2}{\mathbf{c}'\mathbf{M}^{-1}\mathbf{c}} = \mathbf{b}'\mathbf{Q}\mathbf{b}$, with the supremum achieved for $\mathbf{c} = \mathbf{Q}\mathbf{b}$.

(b) If $\mathbf{X}$ is an $n \times k$ matrix, $\mathbf{X}'\mathbf{X} = \mathbf{M}$, $\hat{\mathbf{y}} = \mathbf{X}\mathbf{b}$ and $\hat{\mathbf{y}}_1 = \mathbf{X}\mathbf{M}^{-1}\mathbf{Q}\mathbf{b}$, show that $\hat{\mathbf{y}}_1 = P_{V_1}\hat{\mathbf{y}}$, and $\|\hat{\mathbf{y}}_1\|^2 = \mathbf{b}'\mathbf{Q}\mathbf{b}$, where V_1 is the column space of $\mathbf{X}\mathbf{M}^{-1}\mathbf{A}'$.