（榮譽第一）

國立政治大學八十三學年度第一學期學科考試試題

NATIONAL CHENGCHI UNIVERSITY EXAMINATION FORM

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考試科目 Course	線性模式	開課系級 Dept, & Class	博士班	日期 Date, Period	9月16日 第 節	試題編號 Course No.	

1. Consider the model $\underset{\sim}{Y} \sim N_n(\underset{\sim}{\mu}, \sigma^2 I)$, $\underset{\sim}{\mu} \in V$, $\sigma^2 > 0$. We want to test that $\underset{\sim}{\mu} \in W$ where V and W are specified subspaces, $\dim(W) = k$ and $\dim(V) = p$.

Define $F = \dfrac{\|P_{V|W}\hat{\underset{\sim}{\mu}}\|^2}{(p-k)\hat{\sigma}^2}$, $\phi(F) = \begin{cases} 1 & \text{if } F > F^{\alpha}_{p-k,n-p} \\ 0 & \text{if } F \le F^{\alpha}_{p-k,n-p} \end{cases}$

(a) Let $\underset{\sim}{v} \in R^n$ and let $T \subset R^n$ be a subspace. Show that $\sup\limits_{\substack{\underset{\sim}{c} \in T \\ \underset{\sim}{c} \ne \underset{\sim}{0}}} \dfrac{(\underset{\sim}{c}'\underset{\sim}{v})^2}{\|\underset{\sim}{c}\|^2} = \|P_T \underset{\sim}{v}\|^2$.

(b) Show that the set of intervals $\underset{\sim}{d}'\hat{\underset{\sim}{\mu}} - [(p-k)F^{\alpha}_{p-k,n-p}]^{1/2}\|\underset{\sim}{d}\|\hat{\sigma} \le \underset{\sim}{d}'\underset{\sim}{\mu} \le \underset{\sim}{d}'\hat{\underset{\sim}{\mu}} + [(p-k)F^{\alpha}_{p-k,n-p}]^{1/2}\|\underset{\sim}{d}\|\hat{\sigma}$ is a set of $(1-\alpha)$ simultaneous confidence intervals for $\underset{\sim}{d}'\underset{\sim}{\mu}$, $\underset{\sim}{d} \in V|W$.

(c) Show that the hypothesis $\underset{\sim}{\mu} \in W$ is rejected with the F-test defined above iff at one of the simultaneous confidence intervals does not contain 0.

2. Let $Y_{ij} \sim N(\beta_i, \sigma_i^2)$, $i = 1, 2$, $j = 1, 2, \cdots, n_i$ be independent, and let σ_1^2 and σ_2^2 be known.

(a) Show that $U = \dfrac{\bar{Y}_{1+} - \bar{Y}_{2+} - (\beta_1 - \beta_2)}{\left(\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right)^{1/2}} \sim N(0,1)$ where $\bar{Y}_{i+} = \dfrac{1}{n_i}\sum_{j=1}^{n_i} Y_{ij}$, $i = 1, 2$

(b) Derive the likelihood ratio test $Q(Y)$ for the hypothesis $H_2: \beta_1 = \beta_2$ vs. $H_1: \beta_1$ and β_2 arbitrary.

(c) Show that $2\log Q(Y) = U^2$ and that $U^2 \sim \chi^2(1)$.

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考慮線性模式

$$Y = X\beta + \epsilon \cdots\cdots\cdots (1)$$

其中 $Y: n\times 1$, $X: n\times p$, $\beta: p\times 1$, $E(Y) = X\beta$, $cov(Y) = \sigma^2$
且 $rank(X) = p$. 在增加一些迴歸自變數後將模式(1)擴大為

$$\begin{aligned} Y &= X\beta + Z\gamma + \epsilon \\ &= (X \vdots Z)\begin{pmatrix}\beta \\ \gamma\end{pmatrix} + \epsilon \quad \cdots\cdots (2) \\ &= W\delta + \epsilon \end{aligned}$$

其中 $Z: n\times t$, $\gamma: t\times 1$, $rank(Z) = t$, 且 X 中的行向量與 Z 中的行向量線性獨立。令模式(1)中 β 的 LSE 為 $\hat\beta$, 模式(2)中 δ 的 LSE 為 $\hat\delta = \begin{pmatrix}\hat\beta_G \\ \hat\gamma_G\end{pmatrix}$, 且令 $L = (X'X)^{-1}X'Z$, $R = I_n - X(X'X)^{-1}X$
$M = (Z'RZ)^{-1}$.

$\|Y - W\hat\delta\|^2$

$=$

30% (i) 証明: $\hat\beta_G = \hat\beta - L\hat\gamma_G$

30% (ii) 証明: $RSS_G = RSS - \hat\gamma_G' Z'RY$, 且 $RSS_G \le RSS$, 其中 RSS, RSS_G 分別為模式(1), (2)的殘差平方和.

40% (iii) 証明: $cov(\hat\beta_G) = \sigma^2((X'X)^{-1} + LML')$

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Consider the general linear model which we observe

$$\underline{Y} \sim N_n(\underline{\mu}, \sigma^2 I), \quad \underline{\mu} \in V, \sigma^2 > 0$$

where V is a given p-dimensional subspace of R^n. Let W be a subspace of V, $\dim W = k$, $k < p < n$, $V/W = V \cap W^{\perp}$, $P_V \underline{y}$ the projection of $\underline{y}$ on V, $\hat{\underline{\mu}} = P_V \underline{Y}$, $\hat{\sigma}^2 = \|P_{V^\perp}\underline{Y}\|^2/(n-p)$.

Do the following problems:

1. Show that (a) $P_W \underline{y} = P_W(P_V \underline{y}) = P_V(P_W \underline{y})$, (b) $P_{V/W}\underline{y} = P_V \underline{y} - P_W \underline{y}$ (c) $V/(V/W) = W$. (10% each)

2. (a) Let A be a $k \times n$ matrix such that $A P_V A'$ has rank k. Find a $1-\alpha$ confidence region for $A\underline{\mu}$. (10%)
   (b) Find a set of $1-\alpha$ simultaneous confidence intervals for $\langle \underline{d}, \underline{\mu} \rangle$, where $\underline{d} \in V/W$.

3. Let A be a $k \times n$ matrix, show that
   (a) $A\hat{\underline{\mu}}$ is the best linear unbiased estimator of $A\underline{\mu}$ (10%)
   (b) If $B\underline{Y}$ is a linear unbiased estimator of $A\underline{\mu}$, then $\mathrm{Cov}(B\underline{Y}) - \mathrm{Cov}(A\hat{\underline{\mu}})$ is nonnegative definite. (10%)

4. (a) Let $F = \dfrac{\|P_{V/W}\underline{Y}\|^2 (n-p)}{\|P_{V^\perp}\underline{Y}\|^2 (p-k)}$, show that $F \sim F_{p-k,\, n-p}\left(\dfrac{\|P_{V/W}\underline{\mu}\|^2}{\sigma^2}\right)$ (10%)
   (b) a size α test for testing $H_0: \underline{\mu} \in W$, reject H_0 iff $F > F^{\alpha}_{p-k,\, n-p}$ (5%)
   (c) The F in (a) can be rewritten as
   $$F = \frac{\|P_{V/W}\hat{\underline{\mu}}\|^2}{(p-k)\hat{\sigma}^2}$$
   (5%)
   (d) [illegible]