※請將此份試題作答於第一卷。

國立政治大學 統計研究所八十一學年度第一學期資格考試試題

科目：

第一題：

(I) Let $X_1, \ldots, X_n$ be iid $N(\mu, \sigma^2)$ random variables.
(30%)

(i) Consider estimation of σ^2 when μ is unknown.

(a) Derive the mininum risk equivariant estimator of σ^2 under the group of location-scale transformations ($X' = a + bX$, a is real, and $b > 0$) and the loss function $L(d, \theta) = \{(d - \theta)^2/\theta^2\}$.

(b) Compare the performance of this MRE estimator with those of UMVUE (uniform munimum variance unbiased estimator) and MLE (maximum likelihood estimator) by the expected value of $L(d, \theta)$ given in (a).

(ii) If σ^2 is known (given), let the UMVUE of $p = P(X_1 \le t)$ be $\delta_1 = \Phi(\{n/(n-1)\}^{1/2}(t - \overline{X}))$, and let $\delta_2 = n^{-1}\{\text{No. of } X_i \le t\}$. Compute the values of the asymptotic relative efficiency of δ_2 relative to δ_1 in estimating p.

第二題：

(II) Let $(X_1, \ldots, X_s)$ and Y have finite second moments, $\gamma_i = Cov(X_i, Y)$ and Σ be the positive definite covariance matrix of the X_i's .
(20%)

(i) Define the multiple correlation coefficient between Y and the vector $(X_1, \ldots, X_s)$, and give its espression.

(ii) Let $I(\theta)$ be the Fisher information (matrix) for the vector parameter $\theta = (\theta_1, \ldots, \theta_s)$. Assume the family $\{p_\theta(x)\colon \theta \in \Theta$, a convex open region in $\mathbb{R}^s\}$ of pdf has common support in x, and the partial derivatives $\partial p_\theta(x)/\partial\theta_i$, $i = 1, \ldots, s$, exist; also differentiation under integral signs are legitimate, and $I(\theta)$ is positive definite. If $\delta(X)$ is any estimator of a functional $g(\theta)$ with finite second moment. Show that it follows from (i) that

$Var_\theta\delta(X) \ge \alpha' I^{-1}(\theta)\alpha$, where $\alpha' = (\alpha_1, \ldots, \alpha_s)$, with $\alpha_i = \partial E_\theta\{\delta(X)\}/\partial\theta_i$.

國立政治大學 統計 研究所八十一學年度第一學期資格考試試題 ※請將此份試題作答於第二卷。

科目：[illegible]

第三題：

(20%) 設樣本 $X_1, X_2, \ldots, X_n$ 来自 p.d.f 或 p.m.f. 為 $f(\cdot;\theta)$ 之母体，$\theta \in \Theta \subseteq R$. 令 $X = (X_1, \ldots, X_n)$. 設 $\{f(\cdot;\theta) \mid \theta \in \Theta\}$ 對統計量 $T(X)$ 具 MLR (monotone likelihood ratio). 現考慮 $H_0: \theta \le \theta_0$ 對 $H_1: \theta > \theta_0$，其中 θ_0 已知. 令

$$\Phi(x) = \begin{cases} 1, & \text{若 } T(x) > K \\ d, & \text{若 } T(x) = K \\ 0, & \text{若 } T(x) < K \end{cases}$$

其中 d, K 使 Φ 的大小 (size) 為 α.

証明：Φ 是大小為 α 的 UMP 檢定.

第四題：

設樣本 $X_1, X_2, \ldots, X_n$ 来自分配函數(d.f.) 為 F 之母体. 令 $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$ 為順序統計量，$\hat{F}_n$ 為経驗分配函數 (empirical d.f.)，即 $\hat{F}_n(x) = (X_1, \ldots, X_n \text{ 中 } \le x \text{ 的個數})/n$.

(3%) (1) 証明：$P[X_{(k)} > x] = P[n\hat{F}_n(x) \le k-1]$

(8%) (2) 証明：

$$t < F(x) \Rightarrow P[\hat{F}_n(x) \le t] \le (t - F(x))^{-4} \cdot E(\hat{F}_n(x) - F(x))^4 \le (t - F(x))^{-4} \cdot \frac{A}{n^2}$$

$$t > F(x) \Rightarrow P[\hat{F}_n(x) \ge t] \le (t - F(x))^{-4} \cdot \frac{A}{n^2}$$

其中 A 是常數. (可利用下列結果. $X \sim b(n,p) \Rightarrow E(X-np)^4 = 3(npq)^2 + npq(1-6pq)$，而 $q = 1-p$.)

令 $D_n = \sup_x |\hat{F}_n(x) - F(x)|$. 又設 F 是區間 $(0,1)$ 上均匀分配之 d.f.

(8%) (3) 証明：$D_n = \max\left\{ \max_{1 \le k \le n} \left| X_{(k)} - \frac{k}{n} \right|, \max_{1 \le k \le n} \left| X_{(k)} - \frac{k-1}{n} \right| \right\}$

(8%) (4) 証明：$P\left[\max_{1 \le k \le n} \left| X_{(k)} - \frac{k}{n} \right| > \varepsilon \right] \le \frac{C}{n}$，其中 C 是常數.