國立政治大學統計研究所八十學年度第一學期學科考試試題

科目：迴歸分析專題

第一部份： 基本假設和符号

設 $Y = X\beta + \varepsilon$，其中 $X: n\times p$，$\mathrm{rank}(X) = p$，

$\beta = \begin{pmatrix}\beta_0\\ \beta_1\\ \vdots\\ \beta_{p-1}\end{pmatrix} : p\times 1$，

$\varepsilon \sim N_n(0, \sigma^2 I_n)$.

令 $\hat\beta = \begin{pmatrix}\hat\beta_0\\ \hat\beta_1\\ \vdots\\ \hat\beta_{p-1}\end{pmatrix}$ 是 β 的 LSE，$\hat Y = \begin{pmatrix}\hat Y_1\\ \vdots\\ \hat Y_n\end{pmatrix} = X\hat\beta$，

$e = \begin{pmatrix}e_1\\ \vdots\\ e_n\end{pmatrix} = Y - \hat Y$.

一、令 $H = X(X'X)^{-1}X' = (h_{ij})$.

證明 (i) $\sum_{i=1}^{n} h_{ii} = p$，$\forall\ 1\le i\le n$，$h_{ii} = h_{ii}^2 + \sum_{j\ne i} h_{ij}^2$

(12%) (ii) $\forall\ 1\le i\le n$，$0\le h_{ii}\le 1$

(iii) 當 $h_{ii} = 0$ 時，$\hat Y_i = 0$

當 $h_{ii} = 1$ 時，$\hat Y_i = Y_i$

二、若 A, D 對稱且必要的逆矩陣都存在，

則 $\begin{pmatrix} A & B\\ B' & D\end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} + FE^{-1}F' & -FE^{-1}\\ -E^{-1}F' & E^{-1}\end{pmatrix}$

其中 $E = D - B'A^{-1}B$，$F = A^{-1}B$.（不必證明）

利用上述公式做下列的問題.

設 $X = (x_0, x_1, \cdots, x_{p-1}) = (W, x_{p-1})$

證明 (i) $|X'X| = |W'W| \, (x'_{p-1} x_{p-1} - x'_{p-1} W (W'W)^{-1} W' x_{p-1})$

(18%) (ii) $Var[\hat{\beta}_{p-1}] \geq \sigma^2 (x'_{p-1} x_{p-1})$

(iii) 在 (ii) 中等号成立的充要條件為

$\forall\ i = 0, 1, 2, \cdots, p-2, \quad x'_i x_{p-1} = 0$

三. 設 $A: q \times p$, 且 $\text{rank}(A) = q$. 在 $A\beta = C$ 的限制

(10%) 下, 求 β 的 LSE.

四. 設 $\begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix} = \begin{pmatrix} x'_1 \\ \vdots \\ x'_n \end{pmatrix} \beta + \varepsilon$

令 $S^2 = \sum_{i=1}^{n} e_i^2 / n-p$: 殘差均方 (residual mean square)

$S_{(i)}^2$ = 去掉第 i 個資料點 (Y_i, x'_i) 後所得之殘差均方.

若 $h_{ii} \neq 1$, 求 $\dfrac{e_i}{S_{(i)} \sqrt{1-h_{ii}}}$ 的機率分配,

(10%) 其中 h_{ii} 的定義如第一題中.

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第二部份

迴歸分析

一. For the model $\underset{\sim}{y} = X\underset{\sim}{\beta} + \underset{\sim}{\epsilon}$, with full rank of X, and $\underset{\sim}{\epsilon} \sim N(\underset{\sim}{0}, \sigma^2 I_n)$.
Let $\hat{\underset{\sim}{\beta}} = (X'X)^{-1}X'\underset{\sim}{y}$, $S^2 = [\underset{\sim}{y}'\underset{\sim}{y} - \hat{\underset{\sim}{y}}'\hat{\underset{\sim}{y}}]/n-2 = SSE/n-2$, where $\hat{\underset{\sim}{y}} = X\hat{\underset{\sim}{\beta}}$.

a) Show that $\hat{\underset{\sim}{\beta}}$ and S^2 are joint sufficient statistics for $\underset{\sim}{\beta}$ and σ^2.

b) Show that $\hat{\underset{\sim}{\beta}}$ and S^2 are independent.

c) Show that $\hat{\underset{\sim}{\beta}}$ and S^2 are efficient.

二. For the model $\underset{\sim}{y} = X\underset{\sim}{\beta} + \underset{\sim}{\epsilon}$, where X may not have full rank, let $\hat{\underset{\sim}{\beta}}$ be any solution of the normal equations $(X'X)\underset{\sim}{\beta} = X'\underset{\sim}{y}$.

a) Show that $\hat{\underset{\sim}{\beta}}$ minimizes SSE.

b) If $\underset{\sim}{W}'\underset{\sim}{\beta}$ is estimable, then the BLUE of $\underset{\sim}{W}'\underset{\sim}{\beta}$ is $\underset{\sim}{W}'\hat{\underset{\sim}{\beta}}$.

三. For the model $\underset{\sim}{y} = X\underset{\sim}{\beta} + \underset{\sim}{\epsilon}$ with $E(\underset{\sim}{\epsilon}) = \underset{\sim}{0}$, $Var(\underset{\sim}{\epsilon}) = \sigma^2 I_n$, but with no intercept term, let $\hat{\underset{\sim}{\beta}}$ be the estimator obtained by the least squares method.

a) Show that the sum of the elements of the estimated residual vector is not equal to zero. And prove that $X'\hat{\underset{\sim}{\epsilon}} = 0$.

b) Show that $R^2 = 1 - \dfrac{\hat{\underset{\sim}{\epsilon}}'\hat{\underset{\sim}{\epsilon}}}{(\underset{\sim}{y} - \bar{\underset{\sim}{y}})'(\underset{\sim}{y} - \bar{\underset{\sim}{y}})}$ can become negative. For which model this R^2 will be assured to satisfy $0 \leq R^2 \leq 1$.